TOPOLOGICAL MODELS FOR SOME QUADRATIC RATIONAL MAPS

V. TIMORIN

ABSTRACT. Consider a quadratic rational self-map of the Riemann sphere such that one critical point is periodic of period 2, and the other critical point lies on the boundary of its immediate basin of attraction. We will give explicit topological models for all such maps.

1. INTRODUCTION

1.1. **The family V_2.** Consider the set V_2 of holomorphic conjugacy classes of quadratic rational maps that have a super-attracting periodic cycle of period 2 (we follow the notation of Mary Rees). The complement in V_2 to the class of the single map $z \mapsto 1/z^2$ is denoted by $V_{2,0}$. The set $V_{2,0}$ is parameterized by a single complex number. Indeed, for any map f of class $V_{2,0}$, the critical point of period two can be mapped to ∞, its f-image to 0, and the other critical point to -1. Then we obtain a map of the form

$$f_a(z) = \frac{a}{z^2 + 2z}, \quad a \neq 0$$

holomorphically conjugate to f. Thus the set $V_{2,0}$ is identified with $\mathbb{C} - 0$.

The family V_2 is just the second term in the sequence $V_1, V_2, V_3, \dots$, where, by definition, V_n consists of holomorphic conjugacy classes of quadratic rational maps with a periodic critical orbit of period n. Such maps have one "free" critical point, hence each family V_n has complex dimension 1. Note that V_1 is the family of quadratic polynomials, i.e., holomorphic endomorphisms of the Riemann sphere of degree 2 with a fixed critical point at ∞. Any quadratic polynomial is holomorphically conjugate to a map $z \mapsto z^2 + c$. For such map, the "free" critical point is 0. Thus V_1 can be identified with the complex c-plane. The family V_1 is the most studied family in complex dynamics. The main object describing the structure of V_1 is the *Mandelbrot set* M defined as the set of all parameter values c such that the orbit of the critical point 0 is bounded under $z \mapsto z^2 + c$.

Similarly to the case of quadratic polynomials, we can define the set M_2 (an analog of the Mandelbrot set for V_2) as the set of all parameter values a such that the orbit of -1 is bounded under f_a. A conjectural description of the topology of M_2 is given in [24]. In this paper, we deal with maps on the *external boundary* of M_2, i.e. the boundary of the only unbounded component of $\mathbb{C} - M_2$.

In [16], M. Rees studies the parameter plane of V_3, which turns out to be much more complicated than V_2.

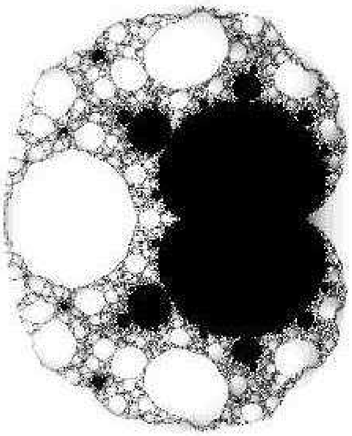


FIGURE 1. The set M_2

1.2. **Invariant laminations.** Invariant laminations were introduced by Thurston [22] to describe quadratic polynomials with locally connected Julia sets. A set L of hyperbolic geodesics in the open unit disk is a *geodesic lamination* if any two different geodesics in L do not intersect, and the union of L is closed with respect to the induced topology on the unit disk. For any pair of points z, w on the unit circle, the geodesic with endpoints z and w will be written as zw. Any geodesic lamination L defines an equivalence relation $\sim_L$ on the unit circle S^1. Namely, two different points on S^1 are equivalent if they are connected by a leaf of L or by a broken line consisting of leaves. For many quadratic polynomials, the Julia set is homeomorphic to the quotient of the unit circle by an equivalence relation $\sim_L$.

We say that a geodesic lamination L on the unit circle is *invariant under the map* $z \mapsto z^2$ if the following conditions hold:

- if $z_1z_2 \in L$, then $z_1^2z_2^2 \in L$,
- if $z_1z_2 \in L$, then $(-z_1)(-z_2) \in L$,
- if $z_1^2z_2^2 \in L$, then $z_1z_2 \in L$ or $z_1(-z_2) \in L$.

Such laminations are also known as *quadratic invariant laminations.* Any quadratic polynomial p defines a quadratic invariant lamination. In many cases, the quotient of the unit circle by the corresponding equivalence relation is homeomorphic to the Julia set J, and the projection of S^1 onto J semi-conjugates the map $z \mapsto z^2$ with the restriction of p to J.

A *gap* of a geodesic lamination is any component of the complement to all leaves in the unit disk. Let L be a quadratic invariant lamination. The map $z \mapsto z^2$ can be extended linearly over all leaves and gaps of L. This extension is called the *lamination map* of L and is denoted by s_L. The image of any leaf under s_L is a leaf or a single point. The image of any gap is a gap, or a leaf, or a single point. Suppose that L is *clean*, i.e. any two adjacent leaves of L are sides of a common finite-sided gap. Then we can also extend the equivalence relation $\sim_L$ to $\mathbb{C}$. The equivalence classes of $\sim_L$ are defined as leaves, finite-sided gaps, or points.

In many cases, the quotient $\mathbb{C}/\sim_L$ is homeomorphic to $\mathbb{C}$. The lamination map s_L defines a continuous self-map $[s_L]$ of this quotient. We say that the lamination L *models* a quadratic polynomial p if the quotient $\mathbb{C}/\sim_L$ is homeomorphic to $\mathbb{C}$, and the map $[s_L]$ is topologically conjugate to p. E.g. any non-parabolic critically finite quadratic polynomial is modeled by the corresponding quadratic invariant lamination. The same is true for many quadratic polynomials with Siegel disks, but not for quadratic polynomials with Cremer points.

Let y_0 be a real number between 0 and 1. Denote by l_0 the diagonal connecting the points $e^{2\pi i y_0}$ and $-e^{2\pi i y_0}$ on the unit circle. Consider all geodesics $z_1 z_2$ in the unit disk such that $z_1^{2^k} z_2^{2^k}$ does not intersect l_0 for all $k < k_0$ and $z_1^{2^{k_0}} z_2^{2^{k_0}} = l_0$, where k_0 is a positive integer depending on $z_1 z_2$. Define the lamination $L(y_0)$ as the closure of the set of all such geodesics. This is a quadratic invariant lamination. If a quadratic polynomial p is modeled by $L(y_0)$, then p belongs to the boundary of the Mandelbrot set. Introduce the following *parameter equivalence* relation on the unit circle. Points $e^{2\pi i y_0}$ and $e^{2\pi i y_0'}$ are parameter equivalent if the laminations $L(y_0)$ and $L(y_0')$ define the same equivalence relation. It turns out that the parameter equivalence relation thus defined also corresponds to a geodesic lamination in the unit disk. This lamination is called the *parameter lamination.* Thurston [22] gave a description of the parameter lamination using his "minor leaf theory". Conjecturally, the boundary of the Mandelbrot set is homeomorphic to the quotient of the unit circle by the parameter equivalence relation. This conjecture is equivalent to the MLC conjecture (stating that the Mandelbrot set is locally connected).

1.3. **Two-sided laminations.** In the theory of quadratic invariant laminations, the single quadratic polynomial $z \mapsto z^2$ is used to build models for the dynamics of many other quadratic polynomials. The Julia set of $z \mapsto z^2$ is the unit circle, and the unit disk is preserved. A similar idea can be used to build models for rational maps of class V_2. To this end, one can use the rational map $z \mapsto 1/z^2$. This is the only map in V_2 not conjugate to a map of the form f_a. Its Julia set is also the unit circle. However, the map $z \mapsto 1/z^2$ interchanges the inside and the outside of the unit disk.

Let us define an analog of quadratic invariant laminations for the map $z \mapsto 1/z^2$. A *two-sided geodesic lamination* is a set of geodesics that live both inside and outside of the unit disk. Note that the outside of the unit disk is also a topological disk in $\overline{\mathbb{C}}$. Geodesics are in the sense of the Poincaré metric (on the inside or on the outside of the unit disk). We will sometimes use $2L$ to denote a two-sided lamination, but this notation does not assume any multiplication by 2 (in other words, $2L$ is to be thought of as a single piece of notation). A two-sided lamination $2L$ gives rise to a pair of laminations L_0 and L_∞, where the leaves of L_0 are inside of the unit circle, and the leaves of L_∞ are outside. The two-sided lamination $2L$ is called *invariant* under $z \mapsto 1/z^2$ if the following conditions hold:

- if $z_1z_2 \in L_0$, then $(1/z_1^2)(1/z_2^2) \in L_\infty$,
- if $z_1z_2 \in L_0$, then $(-z_1)(-z_2) \in L_0$,
- if $z_1^2z_2^2 \in L_0$, then $z_1z_2 \in L_\infty$ or $z_1(-z_2) \in L_\infty$,

and the same conditions with L_0 and L_∞ interchanged. Let $\sim_0$ and $\sim_\infty$ denote the equivalence relations on the unit circle corresponding to the laminations L_0 and L_∞, respectively.

Two-sided laminations were first considered by D. Ahmadi [2]. He used a different language ("laminations on two disks"). In [2], a classification of two-sided laminations is given, similar to the "minor leaf theory" of Thurston [22].

Gaps of two-sided laminations and the corresponding lamination maps are defined in the same way as for geodesic laminations in the unit disk. The equivalence relations $\sim_0$ and $\sim_\infty$ can also be extended to $\overline{\mathbb{C}}$. For a two-sided lamination $2L$, denote by $\sim_{2L}$ the union of the corresponding equivalence relations $\sim_0$ and $\sim_\infty$. We say that a two-sided lamination $2L$ *models* a quadratic rational map $f_a \in V_2$ if the quotient $\overline{\mathbb{C}}/\sim_{2L}$ is homeomorphic to the sphere, and the map $[s_{2L}]$ is topologically conjugate to f_a.

We will now define a particular family of two-sided laminations invariant under $z \mapsto 1/z^2$. Let x_0 be a real number strictly between 0 and 1. Consider the arc σ_0 of the unit circle bounded by the points $e^{2\pi i x_0}$ and $-e^{2\pi i x_0}$ and not containing the point 1. Let σ be any component of the full n-fold preimage of σ_0 under $z \mapsto 1/z^2$. Connect the endpoints of σ by a geodesic in the complement to the unit circle. This geodesic should be inside the unit circle if n is even, and outside if n is odd. For certain values of x_0 (which we will describe explicitly later), the set of geodesics thus constructed is a two-sided lamination. We denote this lamination by $2L(x_0)$. If $2L(x_0)$ exists, then it is clearly invariant under the map $z \mapsto 1/z^2$.

1.4. **Statement of the main theorems.** For a map $f_a \in V_2$, denote by Ω the immediate basin of attraction of the critical cycle $\{0, \infty\}$.

Theorem A. *Suppose that $-1 \in \partial\Omega$. Then the Julia set of f_a is locally connected.*

Let Ω_0 and Ω_∞ denote the components of Ω containing 0 and ∞, respectively. As we will see, the critical point -1 cannot be on the boundary of Ω_∞. Thus, under the assumptions of Theorem A, we can only have $-1 \in \partial\Omega_0$. We will prove in this case that $\overline{\Omega}_0$ is a closed topological disk. Moreover, there is a homeomorphism H of the closed unit disk to $\overline{\Omega}_0$ that conjugates the map $z \mapsto z^2$ with the map $f_a^{\circ 2}$. We say that a point in $\overline{\Omega}_0$ *has angle θ* if this point coincides with $H(re^{2\pi i\theta})$ for some $0 \le r \le 1$.

Theorem B. *Suppose that the critical point -1 belongs to $\partial\Omega_0$ and has angle θ_0. Then, for*

$$x_0 = \sum_{m=1}^{\infty} \frac{[(2^m - 1)\theta_0] + 1}{2^{2m+1}},$$

the two-sided lamination $2L(x_0)$ exists and models the map f_a.

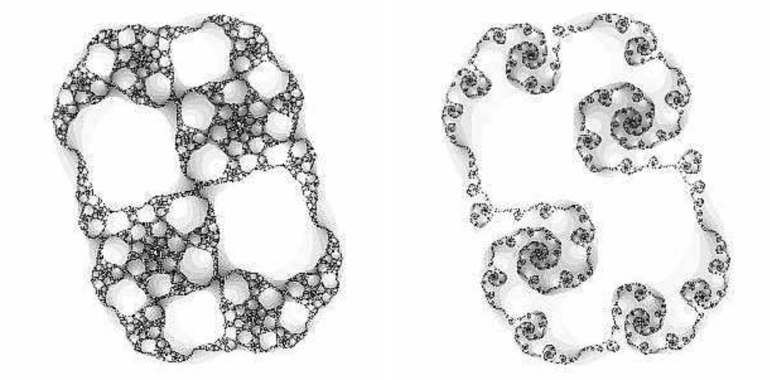


FIGURE 2. The Julia set of $f_a \in V_2$ with $-1 \in \partial\Omega_0$ and of nearby $f_{a'} \in V_2$ with $-1 \in \Omega_0$

The maps f_a from Theorems A and B, together with countably many parabolic maps, form the external boundary of M_2 (the boundary of the unbounded component of $\mathbb{C} - M_2$). We will postpone the proof of this statement to a later publication.

1.5. **Matings.** Consider two quadratic invariant laminations L_0 and L_∞. We can form a two-sided lamination $L_0 \sqcup L_\infty$ by drawing all leaves of L_0 inside the unit circle and all leaves of L_∞ outside the unit circle. The lamination $L_0 \sqcup L_\infty$ is invariant under the map $z \mapsto z^2$ (rather than $z \mapsto 1/z^2$). This lamination is called the *mating* of the laminations L_0 and L_∞. If the quadratic invariant laminations L_0 and L_∞ correspond to quadratic polynomials p_0 and p_∞, and if the lamination $L_0 \sqcup L_\infty$ models a rational map f, then we say that f is a mating of p_0 and p_∞. We write $f = p_0 \sqcup p_\infty$ in this case.

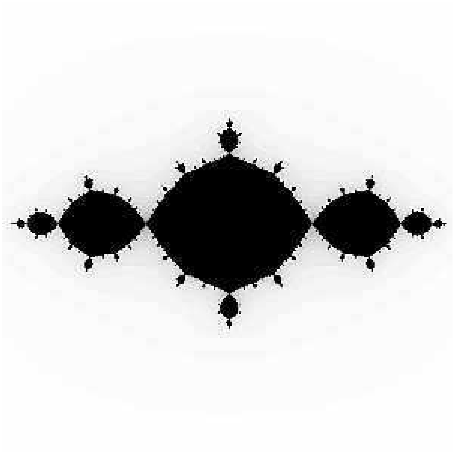
Many maps in V_2 can be described as matings with the quadratic polynomial $z \mapsto z^2 - 1$. The Julia set of this polynomial is called the *basilica*. The dynamics of $z \mapsto z^2 - 1$ can be described by a certain quadratic invariant lamination, which we call the *basilica lamination*. The critical point 0 of the polynomial $z \mapsto z^2 - 1$ is periodic of period two: $f(0) = -1$ and $f(-1) = 0$. Thus $z \mapsto z^2 - 1$ belongs to class V_2. Actually, this is the only polynomial of class V_2.

Theorem B*. *Suppose that the critical point -1 of $f_a \in V_2$ belongs to $\partial\Omega_0$ and has angle θ_0. Let $\theta_0[m]$ denote the m-th binary digit of θ_0. Then, for*

$$y_0 = \frac{1}{3}\left(1 + 3\sum_{m=1}^{\infty} \frac{\theta_0[m]}{4^m}\right),$$

the mating of the basilica lamination and the lamination $L(y_0)$ models the map f_a.

This can be deduced from Theorem B. Actually, the model with a two-sided lamination invariant under $z \mapsto 1/z^2$ is combinatorially equivalent to the mating model. However, the model with a two-sided lamination is simpler in some respects.



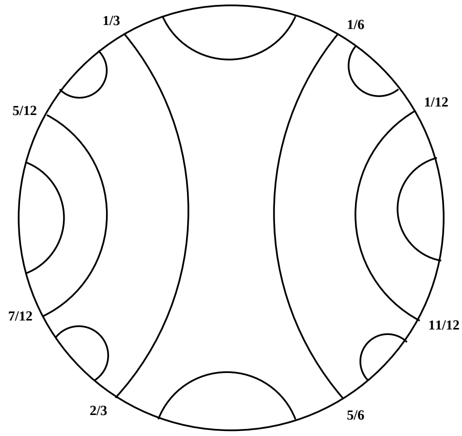


FIGURE 3. The basilica (the Julia set of $z \mapsto z^2 - 1$) and the basilica lamination

For the case, where the critical point -1 is pre-periodic, Theorem A is known, and the proofs of Theorems B and B* are much simpler (they basically follow from the mating criterion given in [20]). In this paper, we will concentrate on the case, where -1 is not pre-periodic. As we will see, the angle θ_0 is irrational in this case.

1.6. **The exterior hyperbolic component.** All theorems we stated so far are about maps on the external boundary of M_2. It is natural to attempt studying topology and dynamics of such maps by approaching them from the *exterior component* $\mathcal{E}$ — the only unbounded component of the complement to M_2. There is a simple dynamical description of the set $\mathcal{E}$: a map $f_a \in V_2$ belongs to $\mathcal{E}$ if and only if the free critical point -1 belongs to the immediate basin of the critical cycle $\{0, \infty\}$. Then we must have $-1 \in \Omega_0$, as we will see.

The Julia set of any map f_a in $\mathcal{E}$ is a quasi-circle, and the restriction of f_a to the Julia set is conjugate to the map $z \mapsto 1/z^2$. This follows from a more general theorem of Sullivan [19]. Thus the topology and the dynamics of the Julia set is the simplest possible. However, a non-trivial combinatorics and a non-trivial dynamics show up when we consider rays for the second iteration $f_a^{\circ 2}$, and how they crash into pre-critical points; more details will come soon.

We give topological models for all maps f_a in $\mathcal{E}$ in terms of Blaschke products. I do not claim any originality here; the point is just to emphasize how general quasi-conformal models of Sullivan and McMullen [9] work for the exterior component of V_2, and to introduce a particular real-analytic identification between $\mathcal{E}$ and the unit disk. The second iteration $f_a^{\circ 2}$ of the map f_a preserves both components of the complement to the Julia set. Pick one particular component. This is an open topological disk. Consider a holomorphic uniformization of this topological disk by the round unit disk. The map corresponding to $f_a^{\circ 2}$ under this uniformization takes the unit disk to itself. Therefore, it is a quartic Blaschke product. It is not hard to see that this Blaschke product must actually be the square of a quadratic Blaschke product

$$B : z \mapsto z\frac{z+b}{\bar{b}z+1},$$

where b belongs to the open unit disk. This gives an idea of how to construct a topological model for f_a.

The unit circle divides the Riemann sphere into two disks — the *inside* and the *outside* of the unit circle. Consider the map $1/B$ that takes the inside to the outside, and the map $1/z^2$ that takes the outside to the inside. We would like to glue these maps together but, unfortunately, they do not match on the boundary. Fortunately, there is a quasi-conformal automorphism Q of the outside of the unit circle such that the maps $Q \circ 1/B$ and $1/z^2 \circ Q^{-1}$ do match on the boundary. They define a global topological ramified self-covering g of the Riemann sphere of degree two. Moreover, there is a natural quasi-conformal structure invariant under g. By the Measurable Riemann Mapping theorem, the ramified self-covering g is topologically conjugate to a quadratic rational map. Clearly, this quadratic rational map must belong to $\mathcal{E}$. Conversely, any map from $\mathcal{E}$ can be obtained by this quasi-conformal surgery.

1.7. **Dynamical rays and external parameter rays.** Let f_a be a map in V_2. The second iteration $f_a^{\circ 2}$ has two super-attracting fixed points 0 and ∞. The other four critical points are -1, the two preimages of -1 under f_a, and the preimage of ∞ under f_a different from 0.

Consider the Green function G for the map $f_a^{\circ 2}$ that is defined by the usual formula

$$G(z) = \lim_{n\to\infty} \frac{\log|f_a^{\circ 2n}(z)|}{2^n}.$$

This function is negative near 0 and positive near ∞. The gradient of G restricted to the complement to the Julia set is a vector field that has singularities at all *pre-critical points* (iterated preimages of critical points). Recall that a *ray* is any trajectory of this vector field.

The α-limit set of any ray is a single pre-critical point, more precisely, an iterated preimage of ∞ or an iterated preimage of -1. The ω-limit set is either a pre-critical point or a point of the Julia set. If the ω-limit set is a pre-critical point, then this point is necessarily an iterated preimage of -1 (because it can not be an iterated preimage of ∞). Consider any iterated preimage z of -1. The point z is a saddle point of the Green function. Thus there are only two rays emanating from z and only two rays crashing into z. The union of the two rays emanating from z, together with the point z itself, is called the *ray leaf centered at z*. Thus the ray leaves are in one-to-one correspondence with iterated preimages of -1.

Suppose that f_a belongs to the exterior component $\mathcal{E}$. Then the critical point -1 belongs to Ω_0. Rays emanating from 0 are parameterized by the *angle*. In a small neighborhood of 0, the map $f_a^{\circ 2}$ is holomorphically conjugate to the map $z \mapsto z^2$. Under this local conjugacy, the point 0 is mapped to 0, and germs of rays are mapped to germs of radial segments. By definition, the angle of a ray is defined as the angle

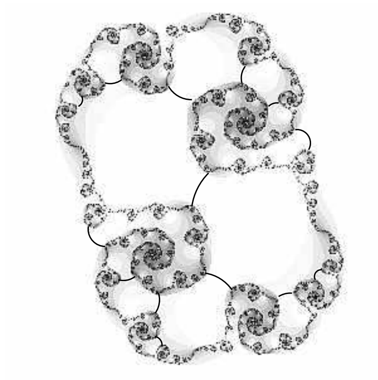


FIGURE 4. Ray leaves for some map in the exterior component of V_2

the corresponding radial segment makes with the real axis. We measure angles in radians/2π. Thus the measure of the full angle is 1. Let $R_0(\theta)$ denote the ray of angle θ emanating from 0. It is not hard to see that there exists a unique ray $R_0(\theta_0)$ that emanates from 0 and crashes into the critical point -1.

Fix an angle θ_0. Consider the set of all parameter values a, for which the ray $R_0(\theta_0)$ crashes into the critical point -1. This set is called the *external parameter ray of angle* θ. We call an external parameter ray periodic or non-periodic according to whether its angle is periodic or non-periodic under the doubling map modulo 1.

M. Rees [15] proved that periodic external parameter rays (except for the zero ray) land at parabolic parameter values. It is possible to deduce from Theorem B that all non-periodic external parameter rays land. The exact dynamical relationship between a non-periodic external parameter ray and its landing point is described below.

1.8. **Ray laminations.** Consider a quadratic rational map f_a in the exterior component $\mathcal{E}$. Assume that f_a does not lie on a periodic parameter ray. It can still lie on a strictly pre-periodic parameter ray. Then each ray leaf of f_a is a curve that is closed in the complement to the Julia set. The closure of this curve in the Riemann sphere intersects the Julia set in two points — the *endpoints* of the ray leaf.

Straighten the Julia set to the unit circle, and each ray leaf to a geodesic in the complement to the unit circle. Then we obtain a two-sided geodesic lamination. Since the restriction of the map f_a to the Julia set is conjugate to the map $z \mapsto 1/z^2$, this two-sided lamination is invariant under $z \mapsto 1/z^2$. We will call this lamination the *ray lamination*. Ray laminations can be described explicitly.

Theorem C. *Let $f_a \in V_2$ be a map in the exterior component. Suppose that f_a lies on a non-periodic external parameter ray of angle θ_0. Then the ray lamination for f_a coincides with the two-sided lamination $2L(x_0)$, where*

$$x_0 = \sum_{m=1}^{\infty} \frac{[(2^m - 1)\theta_0] + 1}{2^{2m+1}}.$$

We will see that all maps from the same parameter ray give rise to the same ray lamination. On the other hand, ray laminations corresponding to maps from different parameter rays, are never equivalent, i.e. one lamination cannot be transformed into the other by a self-homeomorphism of the complement to the unit disk.

What happens if we approach the external boundary along a non-periodic parameter ray? The corresponding ray lamination stays the same, but all leaves become shorter and shorter. In the limit, all leaves of the ray lamination shrink to points. Thus the same two-sided lamination serves both as a ray lamination for a map in the exterior component and as a lamination modeling a map on the external boundary. This picture was the initial motivation for Theorem B stated above. However, the formal proof goes differently. The shrinking of ray leaves can be proved a posteriori, using theorem B.

1.9. **A blow-up of $z \mapsto z^2$.** The explicit formula for x_0 in terms of θ_0 used in Theorems B and C may look mysterious. We will now explain this formula by describing a simple topological construction it comes from.

Let z_0 be any point on the unit circle. There is a unique probability measure μ on the unit circle with the following properties:

- The measure μ is supported on countably many points, namely, on all iterated preimages of z_0 under the map $z \mapsto z^2$ (the point z_0 itself is also regarded as an iterated preimage of z_0).
- For any point z on the unit circle different from z_0, we have $\mu\{z^2\} = 4\mu\{z\}$.

The measure μ can be given by the following formula

$$\mu\{z\} = \sum_{m:\ z^{2^m}=z_0} \frac{1}{2\cdot 4^m}.$$

The summation is over all nonnegative integers m such that $z^{2^m} = z_0$. In particular, if the point z_0 is not periodic under the map $z \mapsto z^2$, then there is at most one summand. The definition of μ can be made simple in the non-periodic case: any preimage of z_0 under the map $z \mapsto z^{2^m}$ has measure $\frac{1}{2\cdot 4^m}$.

It is classically known that there is a unique continuous map $h : S^1 \to S^1$ with the following properties:

- $h(1) = 1$, and 1 is in the center of $h^{-1}(1)$.
- the push-forward of the uniform probability measure under the map h is the measure μ,
- the map h has topological degree 1.

The map h blows up all iterated preimages of the point z_0 under $z \mapsto z^2$ in the following sense. For any point z such that $z^{2^m} = z_0$, the full preimage of z under h is an arc of length $\mu\{z\}$. In particular, the full preimage $h^{-1}(z_0)$ is a half-circle. The following proposition is verified by a simple direct computation:

Proposition 1.1. *If $z_0 = e^{2\pi i\theta_0}$ is not periodic under the squaring map $z \mapsto z^2$, then the half-circle $h^{-1}(z_0)$ is bounded by $e^{2\pi i x_0}$ and $-e^{2\pi i x_0}$, where x_0 is expressed in terms of θ_0 by the formula from Theorems B and C.*

1.10. **Acknowledgements.** I am grateful to M. Lyubich for introducing me to the field of holomorphic dynamics, for his help and encouragement. I had very useful conversations with M. Aspenberg, S. Bonnot, A. Epstein, L. de Marco, H. Hakobyan, M. Rees and M. Yampolsky, to whom I owe my thanks.

2. Two-sided laminations $2L(x_0)$

In this section, we will give details on the explicit construction of two-sided laminations that appear in Theorems B and C. Actually, the construction will be slightly more general, including the two-sided laminations for parabolic maps, not considered in this paper.

2.1. **Formulas for x_0.** Recall that, for a real number θ_0 between 0 and 1 that is not an odd denominator rational number, we defined the corresponding real number x_0 by the formula

$$x_0 = \sum_{m=1}^{\infty} \frac{[(2^m - 1)\theta_0] + 1}{2 \cdot 4^m}.$$

In this subsection, we will find the binary expansion of x_0. Define the functions ν_m on real numbers between 0 and 1 as follows:

$$\nu_m(\theta) = \begin{cases} 0, & \{2^m\theta\} < \theta \\ 1, & \{2^m\theta\} \geq \theta \end{cases}$$

Proposition 2.1. *For any real number θ between 0 and 1, we have*

$$1 + [(2^m - 1)\theta] = [2^m\theta] + \nu_m(\theta).$$

Proof. There are two cases: $[2^m\theta] = [(2^m - 1)\theta]$ and $[2^m\theta] = [(2^m - 1)\theta] + 1$. In the first case, subtracting θ from $2^m\theta$ does not change the integer part, therefore, $\{2^m\theta\} > \theta$, and $\nu_m(\theta) = 1$. In the second case, subtracting θ from $2^m\theta$ changes the integer part, therefore, $\{2^m\theta\} < \theta$, and $\nu_m(\theta) = 0$. □

We can now rewrite the formula for x_0 as follows:

$$x_0 = \sum_{m=1}^{\infty} \frac{[2^m\theta_0]}{2^{2m+1}} + \sum_{m=1}^{\infty} \frac{\nu_m(\theta_0)}{2^{2m+1}}.$$

Let us compute the first sum:

Proposition 2.2. *Let $\theta_0[m]$ denote the m-th digit in the binary expansion of θ_0. Then*

$$\sum_{m=1}^{\infty} \frac{[2^m\theta_0]}{2^{2m+1}} = \sum_{m=1}^{\infty} \frac{\theta_0[m]}{2^{2m}}.$$

Proof. Denote by X the left hand side of this equality. Note that the m-th binary digit of a real number θ is equal to $[2^m\theta] - 2[2^{m-1}\theta]$ for $m \geq 1$. Therefore, the right hand side is

$$\sum_{m=1}^{\infty} \frac{[2^m\theta_0] - 2[2^{m-1}\theta_0]}{2^{2m}} = 2X - X = X. \qquad \square$$

We have proved that

$$x_0 = \sum_{m=1}^{\infty} \frac{\theta_0[m]}{2^{2m}} + \sum_{m=1}^{\infty} \frac{\nu_m(\theta_0)}{2^{2m+1}}.$$

This series represents the binary expansion of x_0. Therefore, we have

Proposition 2.3. *Let $x_0[m]$ denote the m-th binary digit of x_0. Then*

$$x_0[2m] = \theta_0[m], \quad x_0[2m+1] = \nu_m(\theta_0)$$

2.2. **A forward invariant lamination.** Fix a point $z_0 = e^{2\pi i\theta_0}$ on the unit circle. Define a lamination L_0 as follows. We first define a probability measure μ on the unit circle. It is given by the following formula:

$$\mu\{z\} = \sum_{m:\ z^{2^m}=z_0} \frac{1}{2\cdot 4^m}.$$

Next, we consider the map h with the following properties:

- $h(1) = 1$, and 1 is in the center of $h^{-1}(1)$.
- the push-forward of the uniform probability measure under the map h is the measure μ,
- the map h has topological degree 1.

It blows up all iterated preimages of z_0. We connect two points on the unit circle by a geodesic if these two points bound the full preimage of a single point under h. The lamination L_0 is the set of all such geodesics. As we will prove shortly, this lamination is *forward invariant under $x \mapsto x^4$*: for any leaf xy of L_0, either $x^4 = y^4$, or the geodesic x^4y^4 is also a leaf of L_0.

Note that in the definition of the lamination L_0, each leaf $l \in L_0$ comes together with a specific arc subtended by l. Namely, for a leaf xy, the corresponding arc is the full preimage of the point $h(x) = h(y)$ under the map h. We will call this arc the *shadow of the leaf l*. Shadows of different leaves in L_0 do not intersect. Given an arc σ on the unit circle, define *the bridge over σ* as the geodesic connecting the boundary points of the arc σ. Thus the bridge over the shadow of a leaf $l \in L_0$ is this leaf l itself. Denote by l_0 the leaf, whose shadow σ_0 is $h^{-1}(z_0)$.

The lamination L_0 has a distinguished gap G_0 such that all leaves of L_0 are on the boundary of G_0.

Proposition 2.4. *The lamination L_0 defined above is forward invariant under the map $x \mapsto x^4$. Moreover, the map h semi-conjugates the endomorphism $x \mapsto x^4$ of* *the unit circle with the endomorphism $z \mapsto z^2$ everywhere except on the arc σ_0. In other words, $h(x^4) = h(x)^2$ for any point x on the unit circle such that $h(x) \neq z_0$.*

Proof. We first define an endomorphism φ of the unit circle such that L is forward invariant under φ, and then prove that φ is the map $x \mapsto x^4$.

Suppose first that a point x on the unit circle does not belong to a shadow of a leaf of L_0. Then the point $h(x)^2$ has a unique preimage under the map h. Define $\varphi(x)$ to be this preimage. The map φ thus defined admits a continuous extension that maps the full h-preimage of any point z on the unit circle to the full h-preimage of the point z^2, except for $z = z_0$. To fix one such extension, we require that on each arc that is the full h-preimage of some point, the map φ act linearly with respect to the arc-length. Then φ is well-defined everywhere except on σ_0, and the restriction of φ to the full h-preimage of any point on the unit circle multiplies all arc lengths by 4. Indeed, the length of the arc $h^{-1}(z^2)$ is four times bigger that the length of the arc $h^{-1}(z)$, provided that $z \neq z_0$. We can also say where φ should map the arc σ_0 in order to be a self-covering of the unit circle.

In the case, where z_0 is not periodic under $z \mapsto z^2$, the arc σ_0 has length $1/2$. It should be wrapped twice around the circle under the endomorphism φ. Both endpoints of σ_0 should be mapped to the h-preimage of z_0^2, which is a single point. Of course, we require that φ act linearly on σ_0.

In the case, where z_0 is periodic with the minimal period p under the map $z \mapsto z^2$, the orbit of the arc σ_0 under the map $z \mapsto z^4$ consists of p arcs of the following lengths:

$$\frac{4}{2(4^p-1)}, \frac{4^2}{2(4^p-1)}, \dots, \frac{4^p}{2(4^p-1)},$$

the biggest length being that of σ_0. We can arrange that σ_0 wraps more than twice but less than three times around the unit disk under the map φ so that the ends of σ_0 map to the ends of the segment of length $4/2(4^p-1)$ (this segment being covered 3 times by parts of σ_0 under the map φ). In all cases, we can arrange that all arc-lengths in σ_0 get 4 times bigger modulo $\mathbb{Z}$ under the map φ.

We defined a continuous self-map φ of the unit circle that is semi-conjugate to $z \mapsto z^2$ on the complement to the arc σ_0. The semi-conjugacy is establishes by h. It is not hard to see that φ is a self-covering of the unit circle and that $\varphi(1) = 1$. By definition, the lamination L_0 is forward invariant under the map φ.

We will now prove that the map φ just defined multiplies all arc-lengths by 4 modulo $\mathbb{Z}$ (in other words, it multiplies all small arc-lengths exactly by 4). Consider any arc σ on the unit circle, whose length is smaller than $1/4$. We want to show that the length of the arc $\varphi(\sigma)$ is 4 times bigger than the length of the arc σ. Since on each arc of the form $h^{-1}(z)$, the map φ multiplies all arc-lengths by 4, it suffices to assume that σ is the full preimage of the arc $h(\sigma)$ under h. By definition of the measure μ, we have $\mu(h(\sigma)^2) = 4\mu(h(\sigma))$. We also know that $\mu(h(\sigma)^2)$ coincides with the length of the arc $\varphi(\sigma)$. This implies that the length of $\varphi(\sigma)$ is 4 times bigger than the length of σ.

Since the map φ multiplies all arc-lengths by 4 and fixes 1, it must have the form $x \mapsto x^4$. □

2.3. **An invariant lamination.** In this section, we extend the lamination L_0 to a lamination L invariant under the map $x \mapsto x^4$ in the sense of Thurston. Recall that a geodesic lamination in the unit disk is said to be *invariant* under the map $x \mapsto x^d$ if

- it is forward invariant,
- it is *backward invariant*: for any leaf xy of the lamination, there exists a collection of d disjoint leaves, each connecting a preimage of x with a preimage of y under the map $x \mapsto x^d$.
- it is *gap invariant*: for any gap G, the convex hull G' of the image of $\overline{G} \cap S^1$ is a gap, or a leaf, or a single point.

By a *pullback* of a connected set under a continuous map, we mean a connected component of an iterated preimage of this set. Recall that the arc σ_0 was defined as the full preimage of the point z_0 under the map h. The arc σ_0 is the shadow of some leaf l_0. It is easy to see that the shadow of any other leaf in L_0 is a certain pullback of σ_0 under the map $x \mapsto x^4$.

Proposition 2.5. *Consider the set A of all pullbacks of the arc σ_0 under the map $x \mapsto x^4$. The bridges over any two arcs from A are disjoint.*

We need the following lemma:

Lemma 2.6. *Consider two different pullbacks σ and σ' of the arc σ_0 different from σ_0. If the bridges over σ and σ' intersect, then so do the bridges over their images under the map $x \mapsto x^4$, unless σ or σ' coincides with σ_0.*

Proof. If the bridges over σ and σ' intersect, then these arcs intersect each other, but none of them contains the other. The union σ'' of the two arcs is also an arc. If we can show that the length of σ'' is less than $1/4$, then we would conclude that the map $z \mapsto z^4$ acts homeomorphically on σ'', and hence the images of σ and σ' have intersecting bridges.

By the *depth* of a pullback of σ_0 we mean the minimal number n such that σ_0 is the image of the pullback under $x \mapsto x^{4^n}$. The arcs σ and σ' cannot be pullbacks of σ_0 of the same depth, because different pullbacks of the same depth are disjoint. By our assumption, neither of the arcs σ, σ' coincides with σ_0. Then the length of one arc is at most

$$\frac{1}{2}\left(\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}\cdots\right),$$

while the length of the other arc is at most

$$\frac{1}{2}\left(\frac{1}{4^2}+\frac{1}{4^3}+\dots\right).$$

The length of σ'' is thus at most

$$\frac{1}{8} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots < \frac{1}{4}.$$

This proves the lemma. □

Define the set A_0 as the set of all arcs that are shadows of leaves of L_0.

Lemma 2.7. *The union of the set A_0 is backward invariant. In other words, any pullback of any arc in the set A_0 is a subset of some arc in A_0.*

Indeed, this follows from the proof of Proposition 2.4.

Proof of Proposition 2.5. Suppose that there are two arcs from A such that their bridges intersect. Then, applying to this pair of arcs a suitable iterate of the map $x \mapsto x^4$, we can make one of the arcs be σ_0.

Thus we have a pullback σ of the arc σ_0 such that the bridges over σ_0 and σ intersect. But this contradicts Lemma 2.7. □

We can now define a lamination L as the set of bridges over all pullbacks of the arc σ_0. By Proposition 2.5, the leaves of L are disjoint, so that L is indeed a lamination. It is not hard to see that the lamination L does not have any accumulation points inside the unit disk.

Proposition 2.8. *The lamination L is invariant under the self-map $x \mapsto x^4$ of the unit circle.*

Proof. We have already proved forward and backward invariance. It remains only to prove the gap invariance. Define the *span* $P(l)$ *of a leaf* $l \in L$ as the open topological disk bounded by l and the shadow of l. Any gap of L different from G_0 can be described as the complement in a span $P(l)$ to the closures of all spans that lie in $P(l)$. Denote by $G(l)$ the gap associated with the leaf l in this way.

Suppose that l is a leaf of L different from l_0. Then the image of l under the map $x \mapsto x^4$ is another leaf l', and the the gap $G(l)$ maps to the gap $G(l')$ in the following sense: the intersection $\overline{G(l)} \cap S^1$ maps to the intersection $\overline{G(l')} \cap S^1$. Clearly, the gap G_0 maps to itself under the map $x \mapsto x^4$ in this sense. Moreover, G_0 is a *critical gap of degree two*: $\partial G_0/l_0$ maps to ∂G_0 as a topological covering of degree two, if we extend the map $x \mapsto x^4$ linearly over leaves.

It remains to consider the gap $G(l_0)$. This gap is mapped to G_0, and this is also a critical gap. To see that, it is enough to understand what happens with the arc σ_0, but this was described in the proof of Proposition 2.4. □

2.4. **A two-sided lamination.** In this subsection, we extend the lamination L to a two-sided lamination $2L$ invariant under the map $x \mapsto 1/x^2$. By Proposition 1.1, it will be clear that $2L = 2L(x_0)$. In particular, the lamination $2L(x_0)$ exists.

Proposition 2.9. *The lamination L is invariant under the antipodal map $x \mapsto -x$.*

Proof. Indeed, if the shadow σ of some leaf $l \in L$ is a pullback of the arc σ_0 under the map $x \mapsto x^4$, then $-\sigma$ is also a pullback of σ_0. Thus leaves of L map to leaves under the map $x \mapsto -x$, and, clearly, gaps map to gaps. □

Consider the set L' of geodesics outside of the unit circle connecting pairs of points $1/x^2$ and $1/y^2$, where x and y are endpoints of a leaf in L.

Proposition 2.10. *The set L' is a geodesic lamination outside of the unit circle.*

Indeed, by Proposition 2.9, the images of different leaves from L are either the same or disjoint.

We can now consider the two-sided lamination $2L$ that is the union of the inside lamination L and the outside lamination L'. By Proposition 1.1, we have $2L = 2L(x_0)$.

3. The exterior component

In this section, we describe maps in the exterior component $\mathcal{E}$ in terms of a special quasi-conformal surgery performed on Blaschke products. We also discuss combinatorics of rays.

3.1. **Cross-matings of Blaschke products.** Let Δ_0 denote the inside of the unit circle, and Δ_∞ the outside of the unit circle (i.e. the complement to the closed unit disk in the Riemann sphere). The closures of the open disks Δ_0 and Δ_∞ are denoted by $\overline{\Delta}_0$ and $\overline{\Delta}_\infty$, respectively.

A *(finite) Blaschke product* is a product of any finite number of holomorphic automorphisms of the unit disk. The product here is in the sense of multiplication of complex numbers. Any holomorphic automorphism of the unit disk extends to a holomorphic automorphism of the Riemann sphere. Therefore, Blaschke products are also defined on the whole Riemann sphere.

Consider two Blaschke products B_0 and B_∞ of the same degree d. We will make the following assumption on B_0 and B_1: *the restrictions of these maps to the unit circle are expanding in the usual metric.* In particular, this implies that both maps B_0 and B_1 are hyperbolic. Let α_0 be the restriction of the map $1/B_0$ to the unit circle. This map takes the unit circle to itself. Moreover, this is an orientation-reversing self-covering of the unit circle of degree $-d$ (the negative sign represents the change of orientation). The restriction α_∞ of the map $1/B_\infty$ to the unit circle satisfies the same properties.

From a classical theorem of M. Shub [17] it follows that any expanding endomorphism of the unit circle is topologically conjugate to a map $z \mapsto z^k$; the conjugating homeomorphism is unique (see e.g. [5]). In particular, the maps α_0 and α_∞ are topologically conjugate to the map $z \mapsto z^{-d}$. Since α_0 and α_∞ are C^∞, by [18], the conjugating homeomorphism is quasi-symmetric.

The following statement is classical, but we give a proof for completeness:

Lemma 3.1. *Consider two endomorphisms of the unit circle, one of which is expanding. If these two maps have the same topological degree and if they commute, then they coincide.*

Proof. The expanding map is conjugate to the map $z \mapsto z^k$ for some $k \neq 0, \pm 1$. If we lift this map to the universal covering of the unit circle (i.e. to the real line), then we obtain just the linear map $x \mapsto kx$. Assume that another map of topological degree k commutes with $z \mapsto z^k$. The lift of this map to the universal covering has the form $x \mapsto kx + P(x)$, where P is a periodic function. Since the two maps commute, we have

$$(kx) + P(kx) = k(x + P(x)).$$

Therefore, $kP(x) = P(kx)$, and then $k^n P(x) = P(k^n x)$ for all n. The function P is periodic, hence bounded. It follows that

$$P(x) = \lim_{n \to \infty} \frac{1}{k^n} P(k^n x) = 0$$

for all x. □

Let φ denote the self-homeomorphism of the unit circle that conjugates $\alpha_0 \circ \alpha_\infty$ with $\alpha_\infty \circ \alpha_0$. Then we have

$$\varphi \circ \alpha_0 \circ \alpha_\infty \circ \varphi^{-1} = \alpha_\infty \circ \alpha_0.$$

From this equation it follows that the maps $\varphi \circ \alpha_0$ and $\alpha_\infty \circ \varphi^{-1}$ commute. By Lemma 3.1, this is only possible when

$$\varphi \circ \alpha_0 = \alpha_\infty \circ \varphi^{-1}.$$

This is an important functional equation on φ that we will use.

There is a quasi-conformal self-homeomorphism Q of the disk $\overline{\Delta}_\infty$ that restricts to the map φ on the unit circle. This is because φ is quasi-symmetric: any quasi-symmetric automorphism of the unit circle extends to a quasi-conformal automorphism of the unit disk, see [1].

Define a self-map F of the unit sphere as follows. On the disk $\overline{\Delta}_0$, we set F to be $Q \circ (1/B_0)$. On the disk $\overline{\Delta}_\infty$, we set F to be $(1/B_\infty) \circ Q^{-1}$. These two maps match on the unit circle by the functional equation on φ.

There is a quasi-conformal structure on the Riemann sphere that is invariant under the map F. Indeed, we can define this structure to be the standard conformal structure on the unit disk Δ_0, and the push-forward of the standard conformal structure under Q on the disk Δ_∞.

By the Measurable Riemann Mapping theorem of Ahlfors and Bers (see [1]), there is a self-homeomorphism of the sphere that takes the quasi-conformal structure we defined to the standard conformal structure. Let f be a self-map of the Riemann sphere corresponding to the self-map F under this homeomorphism, and J the image of the unit circle. The map f is a holomorphic self-map of the Riemann sphere with the Julia set J (which is a quasi-circle). It has topological degree d, hence it is a rational function of degree d.

We call the map f the *cross-mating* of the Blaschke products B_0 and B_∞.

3.2. **The exterior component.** In this subsection, we consider one particular example of the general construction introduced above. For the map B_0, we take a quadratic Blaschke product

$$B_0(z) = z\frac{z+b}{\bar{b}z+1}$$

with $|b| < 1$. The origin is a fixed point for this map. The critical points $c_{1,2}$ of B_0 are given by the equation $\bar{b}z^2 + 2z + b = 0$. Since we have $|c_1c_2| = 1$, one of the critical points, say c_1, satisfies $|c_1| \le 1$, while for the other critical point c_2 we have $|c_2| \ge 1$. The exact formula for $c_{1,2}$ is

$$c_{1,2} = \frac{-1 \pm \sqrt{1-|b|^2}}{\bar{b}}.$$

We see that c_1 lies in Δ_0, whereas c_2 lies in Δ_∞ (since $|b| < 1$, it is clear from this formula that points $c_{1,2}$ cannot both lie on the unit circle).

Proposition 3.2. *The restriction of B_0 to the unit circle is expanding.*

Proof. By a theorem of Tischler [23], a Blaschke product B restricts to an expanding endomorphism of the unit circle if and only if λB has a fixed point in Δ_0 for all λ in the unit circle. Clearly, the map B_0 satisfies this condition. □

For the map B_∞, we just take $z \mapsto z^2$ (the restriction of this map to the unit circle is obviously expanding). Let $f = f_{[b]}$ be the cross-mating of the Blaschke products B_0 and B_∞. This is a quadratic rational map. It depends smoothly on b. However, the dependence is not analytic, because the Blaschke product B_0 does not depend analytically on b.

Proposition 3.3. *The map f has a super-attracting cycle of period two.*

Proof. Consider the map F from Subsection 3.1. The image of 0 under F is $Q(\infty)$, and the image of $Q(\infty)$ is 0. Thus $\{0, Q(\infty)\}$ is a periodic cycle of period two for the map F. Moreover, $Q(\infty)$ is a critical point of F, hence this cycle is super-attracting. The map f is quasi-conformally conjugate to F. It follows that f also has a super-attracting cycle of period two. □

This proposition means that f is a map in V_2. In particular, it is holomorphically conjugate to some map of the form

$$f_a : z \mapsto \frac{a}{z^2+2z}.$$

Thus, for any $b \neq 0$ in the open unit disk, there is a unique complex number a such that f_a is holomorphically conjugate to $f_{[b]}$. Recall that $f_{[b]}$ was originally defined only up to a holomorphic conjugacy. We can fix this degree of freedom by setting $f_{[b]} = f_a$. For $b = 0$, we obtain the map $z \mapsto 1/z^2$. This defines a map from the unit disk $|b| < 1$ to the parameter space V_2. We will call this map the *cross-mating* *parameterization.* Actually, it is easy to see that each map $f_{[b]}$ belongs to the exterior component $\mathcal{E}$ (this is because all critical points of $f_{[b]}$ are in the immediate basin of attraction of the super-attracting cycle $\{0, \infty\}$).

Proposition 3.4. *The cross mating parameterization is one-to-one: if maps $f_{[b]}$ and $f_{[b']}$ are holomorphically conjugate, then $b = b'$.*

Proof. Indeed, if $f_{[b]}$ and $f_{[b']}$ are holomorphically conjugate on the Riemann sphere, then the squares of the corresponding quadratic Blaschke products

$$B_0(z) = z\frac{z+b}{\bar{b}z+1}, \quad \text{and} \quad B_0'(z) = z\frac{z+b'}{\overline{b'}z+1}$$

are holomorphically conjugate in the unit disk. Since 0 is the only fixed point for each of the maps B_0^2 and ${B_0'}^2$, a conjugating homeomorphism φ must fix 0. Then φ is just the multiplication by some complex number λ such that $|\lambda| = 1$.

The point $-b$ is the only preimage of 0 under B_0. Similarly, the point $-b'$ is the only preimage of 0 under B_0'. Therefore, we must have $b' = \lambda b$. But then the equation $\lambda B_0^2(z) = {B'}_0^2(\lambda z)$ yields $\lambda = 1$, after all cancelations. In particular, $b = b'$. ∎

Proposition 3.5. *The cross-mating parameterization is onto: any quadratic rational map of class $\mathcal{E}$ is holomorphically conjugate to $f_{[b]}$ for some b.*

Proof. Consider any map $f \in V_2$ in the exterior hyperbolic component $\mathcal{E}$. Conjugate $f^{\circ 2}$ by a Riemann map sending Ω_0 to the unit disk and fixing 0. The result is a holomorphic self-covering g of the unit disk of degree 4 such that 0 is a fixed critical point and a preimage $-b \neq 0$ of 0 is also a critical point. In particular, all preimages of 0 have multiplicity 2, which means that there is a well-defined holomorphic branch of the function $\sqrt{g}$. Denote this branch by B_0. Since $B_0(0) = 0$, we conclude that $z \mapsto B_0(z)/z$ is a holomorphic automorphism of the unit disk that maps $-b$ to 0. Therefore, it must have the form

$$\lambda\frac{z+b}{\bar{b}z+1}.$$

Conjugating g with a suitable rotation around the origin, we can arrange that $\lambda = 1$ (with a different choice of b).

The map $f^{\circ 2}$ is holomorphically conjugate to B_0^2, and hence to $f_{[b]}^{\circ 2}$, on the set Ω_0. More precisely, there is a holomorphic embedding $\varphi_0 : \Omega_0 \to \overline{\mathbb{C}}$ such that $\varphi_0 \circ f^{\circ 2} = f_{[b]}^{\circ 2} \circ \varphi_0$. Moreover, with our choice of φ_0, we have $\varphi_0'(0) = 1$. In particular, the 0-ray of $f^{\circ 2}$ emanating from 0 is mapped to the 0-ray of $f_{[b]}^{\circ 2}$ emanating from 0. Since the Julia set of f is locally connected, we can extend φ_0 to the closure of Ω_0.

The map φ_0 takes the critical point -1 of f to a critical point of $f_{[b]}$. Therefore, there is a unique well-defined holomorphic branch φ_∞ of the function $f_{[b]} \circ \varphi_0 \circ f^{-1}$. This branch is defined on Ω_∞, and the union of this branch with φ_0 conjugates f with $f_{[b]}$ on Ω. The latter is verified by a simple direct computation. The map φ_∞

also extends continuously to the Julia set of f. The restrictions of the maps φ_0 and φ_∞ to the Julia set of f coincide. This is because both maps conjugate f with $f_{[b]}$ on the Julia set, and take the the 0-rays of $f^{\circ 2}$ emanating from 0 and ∞ to the 0-rays of $f^{\circ 2}_{[b]}$ emanating from 0 and ∞, respectively. Here we use the fact that if two endomorphisms of the unit circle conjugate $z \mapsto 1/z^2$ with itself, then they differ by a cubic root of unity. Thus the union of the map φ_0 and φ_∞ is a holomorphic automorphism of the Riemann sphere (hence a Möbius map) that conjugates f with $f_{[b]}$. □

3.3. **Ray dynamics: non-periodic case.** Let $f = f_a \in V_2$ be a map in the exterior component. In this subsection, we will study combinatorics of rays for the map $f^{\circ 2}$.

Consider the ray $R_0 = R_0(\theta_0)$ in Ω_0 that emanates from 0 and crashes into -1. Such ray always exists. Indeed, there is at least one ray emanating from 0 that crashes into a pre-critical point (otherwise, the map $f^{\circ 2}$ would be conjugate to the map $z \mapsto z^2$ everywhere on Ω_0). The pre-critical point this ray crashes into must be an iterated preimage of -1. The image of this ray under the corresponding (necessarily even) iteration of f will be the ray emanating from 0 and crashing into -1.

Suppose that the ray R_0 is not periodic under the map $f^{\circ 2}$ (i.e. no iterated image of R_0 is contained in R_0). This means that the angle θ_0 is not periodic under the doubling. There are exactly two rays R_1 and R_2, whose α-limit set is the critical point -1. The images of these rays under the map $f^{\circ 2}$ coincide and lie on the ray $f^{\circ 2}(R_0)$.

Proposition 3.6. *The rays R_1 and R_2 land in the Julia set.*

Proof. It suffices to prove this for one ray, say, for R_1. First, we need to show that the ray R_1 does not crash into pre-critical points. Assume the contrary: the ω-limit set of R_1 is a pre-critical point x. It is an iterated preimage of -1, so that we can write $f^{\circ 2n}(x) = -1$ for some positive integer n.

The set $f^{\circ 2}(R_1)$ lies on the ray containing $f^{\circ 2}(R_0)$. Therefore, the set $f^{\circ 2n}(R_1)$ lies on the ray containing $f^{\circ 2n}(R_0)$. However, the set $f^{\circ 2n}(R_1)$ has the point -1 in its closure, whereas the ray containing $f^{\circ 2n}(R_0)$ does not (because R_0 is not periodic). A contradiction.

We see that R_1 does not crash into pre-critical points. Therefore, its ω-limit set is a connected subset of the Julia set. If this subset contains more than one point, then it contains an arc (i.e. the preimage of an arc under a homeomorphism between the Julia set and the unit circle). In this case, the ω-limit set of a suitable iterated image of R_1 is the whole Julia set. The iterated images of R_1 belong to the rays containing the iterated images of R_0. Thus the ω-limit set of a ray containing a certain iterated image of R_0 is the Julia set.

Consider two strictly pre-periodic rays R' and R'' of different minimal periods emanating from 0. If R_0 is strictly pre-periodic, we assume additionally that the minimal periods of R' and R'' are different from that of R_0. The rays R' and R'' do not crash into pre-critical points, otherwise their suitable iterated images would belong to the ray R_0, which is not pre-periodic or has a different minimal period. The standard argument of Douady and Hubbard [3] now applies to show that R' and R'' land in the Julia set (so that their ω-limits are single well-defined points different from each other). The closures of the rays R' and R'' divide the closed unit disk into two parts, and the closure of any ray emanating from 0 can only belong to one part . This contradicts the statement that the ω-limit set of a certain ray emanating from 0 is the whole Julia set. □

Proposition 3.7. *Any ray for the map $f^{\circ 2}$ either crashes into an iterated preimage of -1 or lands in the Julia set.*

Proof. Consider any ray R. The α-limit set of this ray is an iterated preimage of 0 or an iterated preimage of -1. Thus we can map R to a ray emanating from 0 or from -1 by a suitable iteration of the map $f^{\circ 2}$. In other terms, we can assume without loss of generality that the ray R emanates from 0 or from -1.

Consider the first case: R emanates from 0. Suppose that R does not crash into a an iterated preimage of -1. Then its ω-limit set is contained in the Julia set. The rest of the proof goes exactly as in Proposition 3.6. In the second case, the ray R must coincide with R_1 or R_2. The result now follows from Proposition 3.6. □

Let φ denote the quasi-symmetric homeomorphism between the unit circle and the Julia set of f that conjugates the map $x \mapsto 1/x^2$ with the map f:

$$f(\varphi(x)) = \varphi(1/x^2), \quad x \in S^1$$

Recall that we defined the two-sided *ray lamination* RL associated with f in the following way: $xy \in RL$ if and only if $\varphi(x)$ and $\varphi(y)$ are the landing points of rays emanating from the same iterated f-preimage of -1. The geodesic xy is drawn inside or outside of the unit circle depending on whether this iterated preimage of -1 belongs to Ω_0 or Ω_∞.

3.4. **Proof of Theorem C.** Consider a map $f \in V_2$ in the exterior component that does not belong to a periodic external parameter ray. Let J denote the Julia set of f. We need to prove that the ray lamination RL coincides with some two-sided lamination $2L(x_0)$ corresponding to a point $z_0 = e^{2\pi i\theta_0}$ on the unit circle that is not periodic under the map $z \mapsto z^2$ (here x_0 is expressed through θ_0 as in Theorems B and C). To this end, we recover the map h of Subsection 2.2 in terms of RL. We will use the homeomorphism $\varphi : S^1 \to J$ from the end of the preceding subsection.

For any iterated preimage z of -1, we defined the *ray leaf* $Rl(z)$ as the union of z and the two rays emanating from z. Define a continuous map $\tilde{h} : S^1 \to S^1$ as follows:

- if $\varphi(e^{2\pi i\theta})$ is the landing point of a ray $R_0(\xi)$, then we set $\tilde{h}(e^{2\pi i\theta}) = e^{2\pi i\xi}$;
- otherwise there is a unique ray $R_0(\xi)$ that splits at a precritical point z and such that $Rl(z) \cup J$ separates 0 from $\varphi(e^{2\pi i\theta})$; we set $\tilde{h}(e^{2\pi i\theta}) = e^{2\pi i\xi}$.

Proposition 3.8. *The map $\tilde{h}$ coincides with the map h from Subsection 2.2, with some choice of the point z_0.*

Proof. We will just check that the map $\tilde{h}$ satisfies all properties of the map h. Since $\varphi(1)$ is the landing point of $R_0(0)$, we have $\tilde{h}(1) = 1$. It is also clear that $\tilde{h}$ has topological degree 1. It only remains to verify that the push-forward of the Lebesgue measure under $\tilde{h}$ is the measure μ corresponding to some point z_0 on the unit circle, as it was defined in Subsection 2.2. We denote by $\tilde{\mu}$ the push-forward of the Lebesgue measure under the map $\tilde{h}$.

Consider the ray leaf $Rl(-1) = \{-1\} \cup R_1 \cup R_2$. The landing points of rays R_1 and R_2 divide the Julia set into two arcs. Choose the arc $\varphi(\tilde{\sigma}_0)$ that is separated from 0 by $Rl(-1)$. The arc $\tilde{\sigma}_0$ of the unit circle has length 1/2 (because the boundary points of $\varphi(\tilde{\sigma}_0)$ are mapped to the same point under f, and hence the boundary points of $\tilde{\sigma}_0$ are mapped to the same point under $x \mapsto 1/x^2$). The image of $\tilde{\sigma}_0$ under $\tilde{h}$ is some point z_0 on the unit circle such that $\tilde{\mu}\{z_0\} = 1/2$. Any ray leaf is an iterated preimage of the leaf $Rl(-1)$. Therefore, the images under $\tilde{h} \circ \varphi^{-1}$ of all arcs in J subtended by ray leaves are points on the unit circle that lie in the backward orbit of z_0 under the map $z \mapsto z^2$. Moreover, if $z^{2^m} = z_0$, then we have $\tilde{\mu}\{z\} = \frac{1}{2 \cdot 4^m}$.

We see that the measure $\tilde{\mu}$ coincides with the measure μ corresponding to the point z_0. Then the map $\tilde{h}$ is also the same as the map h. □

Theorem C follows immediately from this proposition.

4. The condition of critical boundary

In this section, we review or establish some combinatorial properties of maps in the family V_2, with an emphasis to maps satisfying the following condition of *critical boundary*: the critical point -1 belongs to the boundary of Ω.

4.1. Immediate basin of the critical 2-cycle.

Let us first recall the setup. Our main object is the following family of quadratic rational self-maps of the Riemann sphere:

$$f_a(z) = \frac{a}{z^2 + 2z}.$$

Infinity is a periodic critical point of period 2 for all maps in this family. The corresponding orbit is $\{0, \infty\}$. The other critical point is -1. Denote by Ω the immediate basin of attraction of the super-attracting cycle $\{0, \infty\}$. Let Ω_0 and Ω_∞ be connected components of Ω containing 0 and ∞, respectively. The restriction of f_a to Ω_∞ is a 2-fold branched covering of Ω_0. It follows that $f_a^{-1}(\Omega_0) = \Omega_\infty$. We will write simply f instead of f_a whenever this notation is unambiguous. The Julia set of f will be denoted by J.

Proposition 4.1. *The critical point -1 does not belong to the set Ω_∞.*

Proof. If $-1 \in \Omega_\infty$, then all critical points of f belong to the same Fatou component. It is known (see e.g. [11, 15]) that in this case, the Fatou component containing the critical points must be invariant, and the Julia set must be totally disconnected. A contradiction. □

Proposition 4.2. *Both sets Ω_0 and Ω_∞ are topological disks.*

Proof. Consider a small disk U containing the origin. For any positive integer n, define the open set U_n as the component of $f^{-n}(U)$ containing 0 or infinity depending on whether n is even or odd. Since $-1 \notin \Omega_\infty$, each set U_n contains at most one critical point. By the Riemann–Hurwitz formula, if U_n is a topological disk, then U_{n+1} is also a topological disk. Thus all U_n are simply connected.

The set Ω_0 is the union of U_n for all even n. As the union of a nested sequence of simply connected open sets, this set is also simply connected. Similarly, Ω_∞ is simply connected. □

Recall that $R_0(\theta)$ denotes the ray in Ω_0 of angle θ. Similarly, we denote by $R_\infty(\theta)$ the ray in Ω_∞ of angle θ. The following proposition is due to Luo [6]:

Proposition 4.3. *The intersection of $\overline{\Omega}_0$ and $\overline{\Omega}_\infty$ contains a fixed point ω of f that is the landing point of both $R_\infty(0)$ and $R_0(0)$.*

Proof. Consider the landing point ω of the 0-ray in Ω_∞ (recall that all rational rays land). This is a point on the boundary of Ω_∞ that is either a fixed point or a point of period 2. However, the map f has only one orbit of period two, namely, $\{0, \infty\}$. It follows that ω is a fixed point. Since ω belongs to the boundary of Ω_∞, it is also on the boundary of Ω_0. □

It is clear that ω is a repelling fixed point.

4.2. **Basilica components.** Let A be a Fatou component of f that maps eventually to Ω_∞. We call such Fatou components *basilica components*, because they correspond to certain Fatou components of the map $z \mapsto z^2 - 1$. The *depth* of a basilica component A is defined as the minimal number n such that $f^{\circ n}(A) = \Omega_\infty$. For a basilica component A, define the *root point* as the landing point of the ray in A of angle zero. It is easy to see that the root point of a depth n basilica component A always belongs to the boundary of a depth $k < n$ basilica component B such that $n - k$ is odd. Moreover, the root point of A coincides with the landing point of a ray in B of angle

$$\frac{m}{2^{\frac{n-k+1}{2}}},$$

where m is an odd integer. Similarly to the case of quadratic polynomials [3], if an iterated preimage of a repelling periodic point is on the boundary of a basilica component A, then it is the landing point of a ray in A with a rational angle.

Proposition 4.4. *The ray $R_\infty(0)$ is the only ray in Ω_∞ landing at ω.*

The proof is similar to that of the following classical statement about quadratic polynomials: there is only one external ray landing at the β fixed point.

Proposition 4.5. *If A is a basilica component different from Ω_∞, then the fixed point ω is not in the closure of A.*

Proof. Suppose that ω is in the closure of A. Then ω must be the root point of A (because some ray in A must land at ω, and this can only be the ray of angle zero). We can assume that A has the minimal depth among all basilica components with this property. In this case, the root point of A must coincide with the landing point of $R_\infty(m/2^n)$, where m is an odd integer, and n is a positive integer. But this point is different from ω by Proposition 4.4. □

Corollary 4.6. *Suppose that -1 is not an iterated preimage of ω. Then any iterated preimage of ω is on the boundary of exactly two basilica components.*

This statement can be easily reduced to the preceding proposition by using iterations of f.

4.3. **Cells.** From now on, we assume that -1 is on the boundary of Ω. In particular, the open set $f^{-1}(\Omega_\infty)$ does not contain critical points. By the Riemann–Hurwitz theorem, this set consists of two connected components. One of these components is Ω_0. The other component contains the point -2 (recall that $f(-2) = \infty$). Denote this component by Ω_{-2}.

Let C_* be the connected component of $\overline{\mathbb{C}} - \overline{\Omega}$ that contains -2. In this case, $\Omega_{-2} \subseteq C_*$. The open set C_* is called the *main cell*. We define *cells of depth n* as connected components of $f^{-n}(C_*)$. Since no cell contains critical points, there are exactly 2^n cells of depth n. For any cell C of depth n, there is a unique component of $f^{-n}(\Omega_{-2})$ contained in C. This basilica component is called the *kernel* of the cell. Note that if a cell has depth n, then the depth of its kernel is $n+1$. Conversely, for each basilica component A different from Ω_{-2}, there is a unique cell containing A as the kernel. The root point of A is also called the *root of the cell*.

We will use cells to encode the dynamics of f. To this end, the following property is crucial:

Theorem 4.7. *For any infinite nested sequence of cells $C^{(1)} \supset C^{(2)} \supset \dots$, the intersection $\bigcap \overline{C^{(n)}}$ consists of a single point.*

We will prove this theorem in Subsection 5.7. The partition of the Julia set into closures of cells has one major disadvantage: the critical point -1 lies on the boundaries of cells rather than in the interior of a cell. This is the reason why we need another partition. We will use the bubble puzzle of Luo [6].

4.4. **Special paths.** Consider a (finite or infinite) sequence (r_n), in which $r_0 = 0$ or ∞ and for $n > 0$, the element r_n is a binary rational number strictly between 0 and 1. For any such sequence, we define the *special path* $\Gamma(r_0, r_1, \dots)$ as follows. If $r_0 = \infty$, then we start at ∞. Go along the ray in Ω_∞ of angle r_1. The landing point a_0 of this ray belongs to the closure of another basilica component A_0. Moreover, a_0 coincides with the root point of A_0. Go from a_0 to the center of A_0 (i.e. the only point in A_0 that is an iterated preimage of ∞) along the zero ray. Repeating the same construction, we obtain a sequence of points a_n and a sequence of Fatou components A_n such that a_n is the landing point of the zero ray in A_n and, at the same time, of the ray in A_{n-1} of angle r_n. We set $A_{-1} = \Omega_\infty$. If the sequence (r_n) stops at some index n, then we stop at the point a_n or at the center of A_n, depending on a context. If $r_0 = 0$, then we need to perform the same construction starting from 0.

Proposition 4.8. *Any iterated preimage of ∞ can be connected to 0 or ∞ by a special path.*

Proof. Note that the preimage of a special path starting at 0 is a pair of special paths starting at ∞:

$$f^{-1}(\Gamma(0, r_1, r_2, \dots)) = \Gamma(\infty, r_1/2, r_2, \dots) \cup \Gamma(\infty, (r_1 + 1)/2, r_2, \dots).$$

Consider a special path $\Gamma(\infty, r_1, r_2, \dots)$ starting at ∞. The preimage of this path is the union of the special path $\Gamma(0, r_1, r_2, \dots)$ and a path starting at -2. But the latter is a part of $\Gamma(\infty, 1/2, r_1, r_2, \dots)$. We see that the preimage of any special path lies in the union of two special paths.

Using this statement, it is now easy to prove the proposition by induction. □

Note that the intersection of any two special paths is an initial segment of both. The image of a special path starting at 0 is a special path starting at ∞:

$$f(\Gamma(0, r_1, r_2, \dots)) = \Gamma(\infty, r_1, r_2, \dots).$$

The image of a special path starting at ∞ is either a special path starting at 0 or the union of a special path starting at ∞ and the path between 0 and ∞ along the zero rays of Ω_0 and Ω_∞. The latter path will be denoted by $[0, \infty]$. More precisely, we have

$$f(\Gamma(\infty, r_1, r_2, \dots)) = \begin{cases} \Gamma(0, 2r_1, r_2, \dots), & r_1 \neq 1/2, \\ \Gamma(\infty, r_2, \dots) \cup [0, \infty], & r_1 = 1/2. \end{cases}$$

4.5. **The β-fixed point.** Consider the following infinite special path $\Gamma^0 = \Gamma(\infty, 1/2, 1/2, \dots)$. Denote by a_n the end of the finite special path

$$\Gamma(\infty, \underbrace{1/2, \dots, 1/2}_{n+1}).$$

Then the point a_0 belongs to the intersection $\overline{\Omega}_\infty \cap \overline{\Omega}_{-2}$. The segment of Γ^0 between points a_0 and a_1 belongs to the closure of $\Omega_{-2} \subset C_*$. Therefore, a_1 is in the closure of the main cell. Note that the boundary of the main cell belongs to $\overline{\Omega}_0 \cup \overline{\Omega}_\infty$. It follows that a_1 cannot be on the boundary of the main cell, otherwise we get a contradiction with Proposition 4.6. We see that $a_1 \in C_*$. By the same argument, all points a_k are in the main cell. Therefore, starting from the point a_1, the whole path Γ^0 is in the main cell.

Consider an injective continuous map $\gamma : [0, \infty) \to \Gamma^0$ such that $\gamma(k)$ is a_{k+1} for all nonnegative integers k. We have

$$f^{\circ k}(\gamma[k, \infty)) = \gamma[0, \infty)$$

for all positive integers k. By a variant of the Douady–Hubbard–Sullivan landing theorem given in [21], it follows that $\gamma(t)$ converges to a repelling or a parabolic fixed point of f (see also [25] for an application of this landing theorem in another puzzle construction). We denote this fixed point by β.

Proposition 4.9. *The fixed point β is different from ω.*

Proof. Suppose that $\beta = \omega$. Consider a small topological disk D around ω. We can arrange that the boundary of this disk intersect each ray $R_\infty(0)$ and $R_0(0)$ at a single point. Then the union of these rays and ω divides D into two parts. The path Γ^0 lies in one part and is invariant under f. However, the two parts are interchanged under f, because the rays $R_0(0)$ and $R_\infty(0)$ are interchanged. A contradiction. □

The fixed point β is not parabolic because there can be no critical point in its basin (recall that the critical point -1 is assumed to be on the boundary of Ω). Thus β is repelling. Since β is the limit of a path in C_*, it follows that β is in the closure of the main cell. However,

Proposition 4.10. *The fixed point β cannot be on the boundary of Ω.*

Proof. If β belongs to the boundary of Ω_0 or to the boundary of Ω_∞, then it belongs to both. There is a ray in Ω_∞ landing at β. This ray must coincide with $R_\infty(0)$, and, therefore, $\beta = \omega$, a contradiction. □

It now follows that β lies in the main cell.

4.6. **The α-fixed point.** The map f has three fixed points. We already discussed two of them, namely, ω and β. Denote the remaining fixed point by α.

Proposition 4.11. *The point α cannot be on the boundary of Ω.*

Proof. Suppose that α is on the boundary of Ω. Then it belongs to $\overline{\Omega}_0 \cap \overline{\Omega}_\infty$. The only possibility for α is to be a Cremer point (otherwise we have rays of Ω_0 and Ω_∞ landing at α, a contradiction). Note that α is a common boundary point of the domains Ω_0 and Ω_∞, which are invariant under $f^{\circ 2}$. However, from the results of Perez-Marco [13, 14] it follows that no Cremer point can be a common boundary point of two disjoint invariant domains. □

Let V denote the component of the complement to $\overline{\Omega}$ that contains α. We will prove that V is the main cell C_*. Otherwise, V is an invariant Fatou component. Since the complement of V is connected, and V contains the fixed point α, but no critical points, V can only be a Siegel disk. The boundary of V lies in the union of $\overline{\Omega}_0$ and $\overline{\Omega}_\infty$. Since the boundary is connected, the two closed sets $\overline{V} \cap \overline{\Omega}_0$ and $\overline{V} \cap \overline{\Omega}_\infty$ must intersect. Let z be any intersection point. This point cannot be fixed (we already know all fixed points of f), and it cannot have period 2, because f has no orbits of period 2 except for $\{0, \infty\}$. It follows that there are at least 3 different points belonging to the closures of the three sets V, Ω_0 and Ω_∞. However, this contradicts the following topological statement:

Lemma 4.12. *Let A, B and C be disjoint connected open sets in the sphere. The intersection $\overline{A} \cap \overline{B} \cap \overline{C}$ cannot have more than 2 points.*

Proof. Assume the contrary: there are at least 3 different points

$$x, y, z \in \overline{A} \cap \overline{B} \cap \overline{C}.$$

Consider small disjoint disks $U(x)$, $U(y)$ and $U(z)$ around these points. Let us also fix some points $a \in A$, $b \in B$ and $c \in C$. We can connect each of the points a, b and c to each of the disks $U(x)$, $U(y)$ and $U(z)$ by simple paths in A, B or C. We can also arrange that these paths do not intersect in $U(x)$, $U(y)$ and $U(z)$. Thus we have 9 curves that do not intersect except at the endpoints and that connect each of the three points with each of the three disks. But this is impossible, because the complete bipartite graph $K_{3,3}$ is not planar. □

Proposition 4.13. *The fixed points α and β lie in different cells of depth 1.*

Proof. Assume the contrary: both α and β lie in a cell C_0 of depth 1. We have a well-defined holomorphic branch $f^{-1} : C_* \to C_0$. Since $C_0 \subset C_*$ (this is because $\beta \in C_*$), we can iterate this branch. Due to the explicit description of holomorphic dynamics on hyperbolic surfaces (see e.g. [10]), the branch $f^{-1} : C_* \to C_0$ has a unique attracting fixed point β, and all forward orbits under this branch converge to β. In particular, $\alpha = \beta$, a contradiction. □

Denote the cells of depth 1 by C_0 and C_1. From Proposition 4.13 it follows that both C_0 and C_1 are subsets of C_*. By induction, we also conclude that all iterated preimages of the main cell are in the main cell.

4.7. **Topology of $\overline{\Omega}$.** In this subsection, we study the topology of $\overline{\Omega}$. In particular, we prove that both sets $\overline{\Omega}_0$ and $\overline{\Omega}_\infty$ are full (recall that a closed subset of the sphere is *full*, if its complement is connected and simply connected). Let Ω^*_∞ denote the union of $\overline{\Omega}_\infty$ and all components of $\overline{\mathbb{C}} - \overline{\Omega}_\infty$ not containing 0. The set Ω^*_∞ is a full closed set. Similarly, define Ω^*_0 as the union of $\overline{\Omega}_0$ and all connected components of $\overline{\mathbb{C}} - \overline{\Omega}_0$ not containing ∞.

Proposition 4.14. *We have $f(\Omega^*_0) \subseteq \Omega^*_\infty$.*

Proof. Indeed, consider any component V of the complement to $\overline{\Omega}_0$ that does not contain ∞. Suppose that $f(V)$ intersects the component of $\overline{\mathbb{C}} - \overline{\Omega}_\infty$ containing 0. Then V intersects the component of $\overline{\mathbb{C}} - \overline{\Omega}_0$ containing ∞, because ∞ is the only preimage of 0. A contradiction. □

The following topological statements are intuitively obvious, but we give a formal proof:

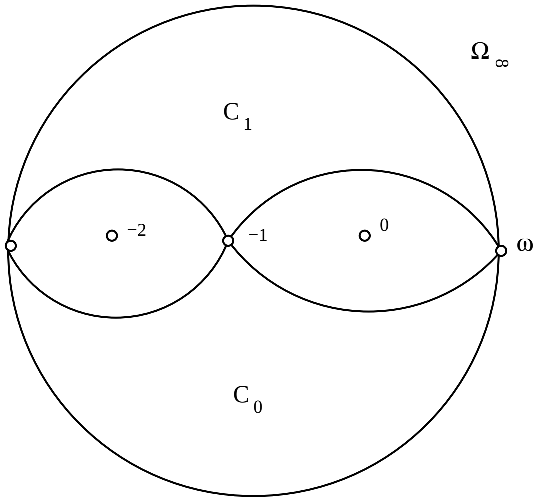


FIGURE 5. The cells C_0 and C_1

Proposition 4.15. *The interiors of* Ω_0^* *and* Ω_∞^* *are disjoint open topological disks.*

Proof. Let U_0 and U_∞ denote the complements to Ω_0^* and Ω_∞^*, respectively. The sets U_0 and U_∞ are open topological disks.

Let us first show that the interior of Ω_0^* is connected. Consider any connected component V of the interior. Then the boundary of V is a subset of ∂U_0. If V does not contain Ω_0, then we also have $\partial V \subseteq \partial\Omega_0$. This contradicts Proposition 4.12 applied to the sets V, U_0 and Ω_0. Since the closure of the connected set U_0 is connected, the interior of Ω_0^* is simply connected.

It remains to prove that Ω_0^* and Ω_∞^* are disjoint. Assume the contrary: there is a component V of the complement to $\overline{\Omega}_0$ that does not contain ∞ and that intersects a component W of $\overline{\mathbb{C}} - \overline{\Omega}_\infty$ not containing 0. It is easy to see that in this case we must have $V = W$. Proposition 4.12 applied to V, Ω_0 and Ω_∞, gives a contradiction. $\square$

The two-valued map f^{-1} takes the interior of Ω_∞^* to the interior of Ω_0^* and to the interior of Ω_{-2}^* (by definition, the set Ω_{-2}^* is the union of all components of the complement to $\overline{\Omega}_{-2}$ that do not contain ∞). Since these sets are disjoint (which can be proved in the same way as in Proposition 4.15), we have two well-defined holomorphic branches of f^{-1} on Ω_∞. In particular, the critical point -1 does not belong to the interior of Ω_0^*.

Proposition 4.16. *The points* 0 *and* -2 *belong to the same component of the complement to* $\overline{\Omega}_\infty$*.*

Proof. Suppose not. In this case, the main cell is a component of the complement to $\overline{\Omega}_\infty$. We know that both fixed points α and β lie in the main cell. There is a well-defined branch of f^{-1} mapping the interior of Ω_∞^* to the main cell (because the main cell contains the interior of Ω_{-2}^*). But the main cell is a subset of Ω_∞^*.

Therefore, we can iterate the considered branch of f^{-1}. It would follow that α and β lie in the same cell of depth 1, a contradiction with Proposition 4.13. $\square$

Proposition 4.17. *The sets* $\overline{\Omega}_\infty$ *and* $\overline{\Omega}_0$ *are full.*

Proof. By Proposition 4.14, we have $f(\Omega_0^*) \subseteq \Omega_\infty^*$. From Proposition 4.16, it follows that $f(\Omega_\infty^*) \subseteq \Omega_0^*$ as well. Therefore, the second iterate of f takes Ω_∞^* to itself. It follows that the interior of Ω_∞^* lies in a single Fatou component. Since this Fatou component intersects Ω_∞, it must coincide with Ω_∞. $\square$

4.8. **Prime end impressions of Ω_∞.** In this subsection, we study the boundary of Ω_∞. Our assumption on the critical point -1 implies that it does not belong to Ω. Then the restriction of $f^{\circ 2}$ to Ω_∞ is holomorphically conjugate to the restriction of $z \mapsto z^2$ to the unit disk. Rays in Ω_∞ correspond to radial segments. Let H denote the biholomorphic map of the open unit disk into Ω_∞ such that $H(x^2) = f^{\circ 2}(H(x))$ for all x with the property $|x| < 1$. Recall that the *prime end impression of angle* θ in $\overline{\Omega}_\infty$ is defined as the set of points $z \in \partial\Omega_\infty$ representable as $\lim_{n\to\infty} H(r_n e^{2\pi i\theta_n})$ for some sequences $\theta_n \to \theta$ and $r_n \to 1$. It is clear that any point on the boundary of Ω_∞ belongs to at least one prime end impression.

Proposition 4.18. *Different prime end impressions of* Ω_∞ *are disjoint.*

Proof. Consider landing points of all binary rational rays in Ω_∞. All these landing points are accessible from outside of $\overline{\Omega}_\infty$ (recall that the complement to $\overline{\Omega}_\infty$ is a topological disk containing 0), because they also belong to boundaries of some basilica components different from Ω_∞. Therefore, the landing points of rays $R_\infty(m/2^n)$ separate the boundary of Ω_∞ into n pieces. Each prime end impression is contained in a single piece. The proposition now follows because we can take n arbitrarily large. $\square$

4.9. **The condition of critical boundary.** Recall that our standing assumption is that the critical point -1 belongs to the boundary of Ω. In this subsection, we will make this condition more specific by showing that -1 cannot lie in $\overline{\Omega}_\infty$:

Proposition 4.19. *The critical point* -1 *does not belong to the boundary of* Ω_∞.

Proof. Assume the contrary: $-1 \in \overline{\Omega}_\infty$. Let θ_∞ be the angle of a prime end impression of Ω_∞ containing -1. Then there is a point x in a small neighborhood of -1 lying on a ray $R_\infty(\theta)$ in Ω_∞, whose angle θ is very close to θ_∞. Consider the point $x' = -2 - x$ symmetric to x with respect to -1. We have $f(x') = f(x)$. Therefore, the point x' lies on the ray $R_\infty(\theta + 1/2)$. Since θ can be made arbitrarily close to θ_∞, we must conclude that -1 belongs to the impression of angle $\theta_\infty + 1/2$. This contradicts Proposition 4.18. $\square$

Since $-1 \notin \overline{\Omega}_\infty$, we must have $-1 \in \partial\Omega_0$.

Proposition 4.20. *We have* $\overline{\Omega}_0 \cap \overline{\Omega}_{-2} = \{-1\}$.

Proof. Take a point $z \in \Omega_0$ very close to -1. Then the point z' symmetric to z with respect to -1 (i.e. $z' = -2 - z$) is also very close to -1, but it belongs to Ω_{-2}. Therefore, -1 is on the boundary of Ω_{-2}.

Suppose now that z_0 is a point in $\overline{\Omega}_0 \cap \overline{\Omega}_{-2}$ different from -1. A small disk around z_0 intersects the union of Ω_0 and Ω_{-2} by two disjoint open sets such that z_0 belongs to the boundaries of both sets. Therefore, a small neighborhood of $f(z_0)$ intersects Ω_∞ by two disjoint open sets containing $f(z_0)$ on their boundaries. It is easy to see that since $\overline{\Omega}_\infty$ is a full set, and Ω_∞ is the interior of this set, such situation is impossible. □

Suppose that the critical point -1 belongs to the prime end impression of angle θ_0 with respect to Ω_0. Then θ_0 is called the *critical angle*.

5. Topological model

In this section, we construct a topological model for maps $f \in V_2$ such that $-1 \in \partial\Omega_0$. We will encode the dynamics of f by *cells*, and use *bubble puzzle pieces* of Luo [6] to prove the convergence of cells.

5.1. The intersection of $\overline{C}_0$ and $\overline{C}_1$.

Recall that C_0 and C_1 are the cells of depth 1. Denote by a_* the landing point of the ray $R_\infty(1/2)$. This point belongs to the boundary of both Ω_∞ and Ω_{-2}. In this subsection, we show that

$$\overline{C}_0 \cap \overline{C}_1 \subseteq \{a_*, -1, \omega\}.$$

It is easy to see that any intersection point of $\overline{C}_0$ and $\overline{C}_1$ belongs to at least two of the following three sets: $\overline{\Omega}_\infty$, $\overline{\Omega}_0$ and $\overline{\Omega}_{-2}$. We already know that the intersection of $\overline{\Omega}_0$ and $\overline{\Omega}_{-2}$ is $\{-1\}$. Therefore, all other intersection points of $\overline{C}_0$ and $\overline{C}_1$ belong to the boundary of Ω_∞. The boundary of Ω_∞ is divided into two parts by the points ω and a_*. Each of the sets $\overline{C}_1 \cap \overline{\Omega}_\infty$ and $\overline{C}_0 \cap \overline{\Omega}_\infty$ belongs to only one part, which can be proved by a simple connectivity argument. But then $\overline{C}_0 \cap \overline{C}_1 \cap \overline{\Omega}_\infty$ is a subset of $\{a_*, \omega\}$. The fact that the set $\overline{C}_0 \cap \overline{C}_1 \cap \overline{\Omega}_\infty$ has at most two points can also be deduced from Proposition 4.12. Actually, we only need this fact.

We know that -1 actually belongs to the intersection $\overline{C}_0 \cap \overline{C}_1$. Later we will see that a_* and ω belong to this intersection as well.

For any point x in the Julia set, whose forward orbit is disjoint with $\{-1, \omega\}$, and any nonnegative integer n, there is a unique cell $C^{(n)}(x)$, whose closure contains x.

5.2. Thickened cells.

Define *thickened cells* $\widehat{C}_0$ and $\widehat{C}_1$ as open topological disks bounded by arcs of small circles around a_*, -1 and ω, arcs of equipotentials in Ω_0, Ω_∞ and Ω_{-2}, and ray segments in Ω_0, Ω_∞ and Ω_{-2}, as in Picture 6. We will assume that $\widehat{C}_0$ and $\widehat{C}_1$ contain all three points a_*, -1 and ω, and that the union $\widehat{C}_0 \cup \widehat{C}_1$ contains the main cell. We can also assume that $C_0 \subset \widehat{C}_0$ and $C_1 \subset \widehat{C}_1$. By a suitable choice of the bounding ray segments we can arrange that $\widehat{C}_0 \cup \widehat{C}_1 \subset f(\widehat{C}_0)$ and $\widehat{C}_0 \cup \widehat{C}_1 \subset f(\widehat{C}_1)$.

Both preimages of -1 belong to the boundary of Ω_∞, but they lie in different thickened cells. One preimage of a_* belongs to the boundary of Ω_0, and the other preimage to the boundary of Ω_{-2}. Let z_0 be the preimage of a_* that lies on the boundary of Ω_0, and z_∞ the preimage of -1 that lies in the same thickened cell as z_0 (then z_∞ and z_0 must be on the boundary of the same component of $f^{-1}(\Omega_{-2})$). To fix the ideas, assume that $z_0, z_\infty \in \widehat{C}_1$.

From Proposition 4.6 it follows that the point z_0 does not belong to $\overline{\Omega_\infty \cup \Omega_{-2}}$. From Proposition 4.19 it follows that the point z_∞ does not belong to $\overline{\Omega_0 \cup \Omega_{-2}}$. Now it is not hard to derive the following

Proposition 5.1. *There is a holomorphic branch $f^{-1} : \widehat{C}_0 \to \widehat{C}_1$ that takes points a_*, -1 and ω to points z_0, z_∞ and ω, respectively, and such that the image of $\widehat{C}_0$ under this branch is compactly contained in $\widehat{C}_1$.*

This branch is defined on $\widehat{C}_0$ rather than on $\widehat{C}_1$, because locally, near the fixed point ω, the branch of f^{-1} fixing ω interchanges $\widehat{C}_1$ with $\widehat{C}_0$. We denote the holomorphic branch $f^{-1} : \widehat{C}_0 \to \widehat{C}_1$ by f_1.

Consider the preimage z_0' of z_∞ that lies on the boundary of Ω_0. Let z_∞' be the preimage of z_0 that shares the boundary of a basilica component with z_0'. We have $z_\infty' \in \overline{\Omega}_\infty$. Clearly, z_0' is disjoint with $\overline{\Omega_\infty \cup \Omega_{-2}}$, and z_∞' is disjoint with $\overline{\Omega_0 \cup \Omega_{-2}}$. Now it is easy to see the following:

Proposition 5.2. *There is a holomorphic branch $f^{-1} : f_1(\widehat{C}_0) \to \widehat{C}_0$ that takes points z_0, z_∞ and ω to points z_∞', z_0' and ω, respectively.*

Denote this branch by f_2. Combining Propositions 5.1 and 5.2, we see that $f_2 \circ f_1$ is a holomorphic branch of f^{-2} defined on $\widehat{C}_0$ such that the image of $\widehat{C}_0$ is compactly contained in $\widehat{C}_0$. In particular, $f_2 \circ f_1$ shrinks all Poincaré distances in $\widehat{C}_0$ by a definite factor.

We can now deduce the convergence of some special nested sequences of cells. Namely, there are two sequences of cells $C_0^{(n)}(\omega)$ and $C_1^{(n)}(\omega)$ uniquely defined by the following properties:

- $C_i^{(n)}(\omega)$ are cells of depth n, and $C_i^{(1)}(\omega) = C_i$ for $i = 0, 1$;
- $C_i^{(n)}(\omega) \subset C_i$ for $i = 0, 1$;
- $f(C_0^{(n+1)}(\omega)) = C_1^{(n)}(\omega)$ and $f(C_1^{(n+1)}(\omega)) = C_0^{(n)}(\omega)$.

The cells $C_0^{(n)}(\omega)$ are uniquely defined by the following reason: $f^{-1}(C_1^{(n-1)}(\omega))$ has two components, one lying in C_0, and the other lying in C_1; the cell $C_0^{(n)}(\omega)$ is the component lying in C_0. Similarly for $C_1^{(n)}(\omega)$. It is easy to see that $C_i^{(n+1)}(\omega) \subset C_i^{(n)}(\omega)$ for $i = 0, 1$.

Proposition 5.3. *We have*

$$\bigcap_{n=1}^{\infty} \overline{C_0^{(n)}(\omega)} = \bigcap_{n=1}^{\infty} \overline{C_1^{(n)}(\omega)} = \{\omega\}.$$

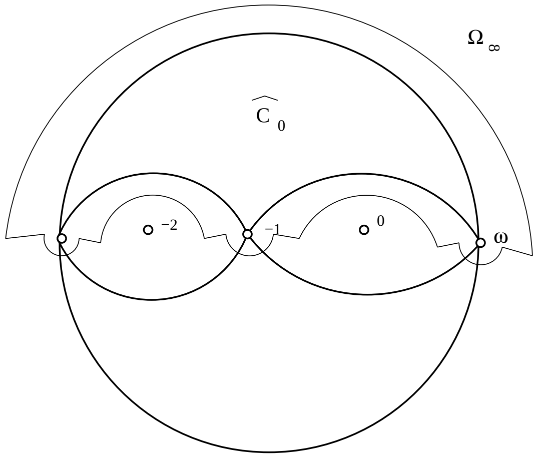


FIGURE 6. The thickened cell $\widehat{C}_0$

Proof. Clearly, it suffices to prove the convergence for the sequence $C_0^{(n)}(\omega)$. The cell $C_0^{(1)}(\omega)$ is contained in the thickened cell $\widehat{C}_0$. It follows by induction that $C_0^{(2n+1)}(\omega)$ is contained in the image of $\widehat{C}_0$ under the n-th iterate of $f_2 \circ f_1$. The proposition now follows from the contraction principle. □

From this proposition, it actually follows that ω belongs to the closures of both cells C_0 and C_1. Then a_*, which is a preimage of ω, also belongs to the closures of both C_0 and C_1.

5.3. **Special paths converging to α.** We will now find some special paths converging to the fixed point α.

Proposition 5.4. *Let $C^{(n)}(\alpha)$ be the cell of depth n containing the fixed point α. Then there is a positive integer n_0 such that the root of $C^{(n_0)}(\alpha)$ belongs to the boundary of Ω_{-2}.*

Proof. Denote by I the intersection of all $\overline{C^{(n)}(\alpha)}$. Suppose that I intersects the boundary of Ω_0 or the boundary of Ω_∞. Since I is forward invariant, it must then intersect both boundaries. There are basilica components intersecting the boundary of Ω_∞ at the landing points of all rays in Ω_∞ with binary rational angles, hence the intersection $I \cap \overline{\Omega}_\infty$ must be in a single prime end impression of Ω_∞. Let θ denote the angle of this impression. Since $f^{\circ 2}$ doubles the angles of all rays in Ω_∞, it follows that $\theta = 0$. We must conclude that $C^{(n)}(\alpha)$ coincides with $C_0^{(n)}(\omega)$ or with $C_1^{(n)}(\omega)$ for all n. We now have a contradiction with Proposition 5.3.

The contradiction shows that I is disjoint with $\overline{\Omega}$. It follows that for large n, the closure of the cell $C^{(n)}(\alpha)$ is disjoint with $\overline{\Omega}$. In particular, the root of $C^{(n)}(\alpha)$ does not always belong to $\overline{\Omega}$. Denote the first such depth n by n_0. Clearly, the root of $C^{(n_0)}(\alpha)$ belongs to $\overline{\Omega}_{-2}$. □

We have $f(C^{(n+1)}(\alpha)) = C^{(n)}(\alpha)$. Let A_n be the kernel of the cell $C^{(n)}(\alpha)$. We also set $C^{(0)}(\alpha) = C_*$ and $A_0 = \Omega_{-2}$. Let a_n denote the landing point of the zero ray in A_n. In particular, $a_0 = a_*$. By Proposition 5.4, there is a number n_0 such that $a_{n_0} \in \overline{\Omega}_{-2}$. Consider a special path $\Gamma(\infty, 1/2, r_2)$ connecting points a_0 and a_{n_0}. We can extend this path to the infinite special path

$$\Gamma_1 = \Gamma(\infty, 1/2, r_2, r_2, \dots)$$

There is a well-defined holomorphic branch g of f^{-n_0} that maps $C^{(0)}(\alpha) = C_*$ to $C^{(n_0)}(\alpha)$. The path Γ_1 is forward invariant under g. Clearly, it converges to the fixed point α.

The map $f^{\circ n_0}$ takes the path Γ_1 to itself (modulo the segment $[0,\infty]$). In this sense, Γ_1 is periodic under f. Denote the period by q. However, Γ_1 is not fixed, because otherwise we would have $r_2 = 1/2$, and Γ_1 would coincide with the special path Γ^0 converging to β. Consider all images of Γ_1 under iterations of f (regarded as special paths starting at ∞ or 0; the segment $[0,\infty]$ appearing in the image should be disregarded), and denote them by $\Gamma_1, \dots, \Gamma_q$, where $\Gamma_i = f^{\circ i-1}(\Gamma_1)$. All paths Γ_i converge to the fixed point α.

We have

$$\Gamma_2 = \Gamma(\infty, r_2, r_2, \dots), \quad \Gamma_3 = \Gamma(0, 2r_2, r_2, r_2, \dots).$$

The union of the special paths Γ_1 and Γ_3 together with α and the segment $[0,\infty]$ is a loop that divides the Riemann sphere into two topological disks. Consider the component of the complement to this loop that contains -1. It also contains either all rays in Ω_∞, whose angles are bigger than $1/2$ or all rays in Ω_∞, whose angles are smaller than $1/2$.

Proposition 5.5. *The critical angle θ_0 is between r_2 and $2r_2$. In particular, $\theta_0 \neq 0$.*

Proof. Consider the special paths Γ'_1 and Γ'_3 symmetric to the special paths Γ_1 and Γ_3 with respect to the critical point -1. We have

$$\Gamma'_1 = \Gamma(0, r_2, r_2, \dots), \quad \Gamma'_3 = \Gamma(\infty, 1/2, 2r_2, r_2, r_2, \dots).$$

Both paths Γ'_1 and Γ'_3 converge to the point $\alpha' = -2-\alpha$ symmetric to α with respect to -1. We see that -1 is contained in a region bounded by parts of the special paths Γ_1, Γ'_1, Γ_3, Γ'_3 together with the points α and α' (this region is bounded away from $\overline{\Omega}_\infty$, see Picture 7). Therefore, the critical angle θ_0 is between r_2 and $2r_2$. □

5.4. **Bubble puzzle.** We use the ideas of Luo [6] to construct an analog of the Yoccoz puzzle for maps on the external boundary. The argument will be specific to our situation. The general construction of puzzles for V_2 (both dynamical and parameter) with application to matings is a work in progress by M. Aspenberg and M. Yampolsky (I am grateful to M. Aspenberg for communicating their ideas). In this subsection and later, we assume that the map f is not critically finite; in other words, the critical point -1 is not pre-periodic under f.

Denote by E_∞ some equipotential curve in Ω_∞ and by E_0 some equipotential curve in Ω_0. Let U be the component of the complement to $E_\infty \cup E_0$ containing -1. By choosing appropriate equipotentials E_∞ and E_0, we can arrange that $f^{-1}(U)$ be compactly contained in U. Puzzle pieces of depth zero are defined as connected components of the complement to the set

$$[0,\infty] \cup \bigcup_{i=1}^{q} \Gamma_i \cup \{\alpha\} \cup E_\infty \cup E_0,$$

intersecting the Julia set. A *puzzle piece* $P^{(n)}$ of any depth n is defined as a connected component of $f^{-n}(P^{(0)})$, where $P^{(0)}$ is a puzzle piece of depth 0. For any point $z \in J$ not on the boundary of a puzzle piece, let $P^{(n)}(z)$ denote the puzzle piece of depth n containing z. Puzzle pieces $P^{(n)}(-1)$ are called *critical puzzle pieces.* According to our assumption, -1 is not pre-periodic, therefore, the critical puzzle pieces are well defined.

Each path Γ_i corresponds to a *bubble ray* — the union of all basilica components intersecting Γ_i. However, we use paths Γ_i instead of the corresponding bubble rays because two different bubble rays may touch at iterated preimages of the critical point -1.

5.5. **An example.** Before discussing general combinatorics of bubble puzzles, let us work out one particular example. Suppose that $r_2 = 1/4$. Then $q = 3$, and the special paths Γ_i, $i = 1, 2, 3$, converging to the fixed point α are

$$\Gamma_1 = \Gamma\left(\infty, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \dots\right), \quad \Gamma_2 = \Gamma\left(\infty, \frac{1}{4}, \frac{1}{4}, \dots\right), \quad \Gamma_3 = \Gamma\left(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \dots\right).$$

Consider also preimages of these paths (or, equivalently, paths symmetric to these paths with respect to -1):

$$\Gamma_1' = \Gamma\left(0, \frac{1}{4}, \frac{1}{4}, \dots\right), \quad \Gamma_2' = \Gamma\left(\infty, \frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \dots\right), \quad \Gamma_3' = \Gamma\left(\infty, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \dots\right).$$

The paths Γ_1', Γ_2' and Γ_3' converge to the point α' symmetric to α with respect to -1, i.e. $\alpha' = -2 - \alpha$. The six paths Γ_i, Γ_j', $i, j = 1, 2, 3$, divide the open set U into 5 pieces (see Picture 7).

We see that no puzzle piece of depth 1 is compactly contained in a puzzle piece of depth 0. Next, we need to look for puzzle pieces of depth 2 compactly contained in puzzle pieces of depth 0. Indeed, there are two puzzle pieces of depth 2 compactly contained in $P^{(0)}(-1)$. They are marked with sign "+". However, one of these two puzzle pieces is still useless (namely, the one that does not intersect Ω_∞), because the critical orbit never enters it.

5.6. **Critical annuli.** The critical annuli for the bubble puzzle are defined in the same way as for the Yoccoz puzzle: the *critical annulus* $R^{(n)}(-1)$ of depth n is $P^{(n-1)}(-1) - \overline{P^{(n)}(-1)}$. If $R^{(n)}(-1)$ is not a topological annulus, then it is called a *degenerate annulus.* We saw that there may be no nondegenerate critical annulus at

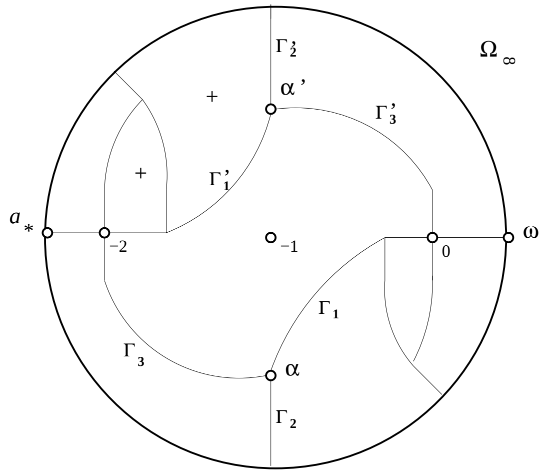


FIGURE 7. The bubble puzzle for $r_2 = 1/4$ (this is a very schematic picture not showing equipotentials and rays in Ω_∞)

all. In this respect, the bubble puzzle is combinatorially different from the Yoccoz puzzle for quadratic polynomials, although the combinatorics of the two puzzles is still very similar.

Recall that for quadratic polynomials, the existence of a nondegenerate critical annulus was settled by the following statement (see [10, 7]): for a non-renormalizable quadratic polynomial, the critical orbit enters a non-critical puzzle piece of depth 1 touching the point $-\alpha$ (where α is the α-fixed point). There is an analog of this statement for the maps under consideration:

Proposition 5.6. *Let α' be the preimage of α different from α, i.e. $\alpha' = -2 - \alpha$. The critical orbit enters a puzzle piece of depth 1 touching α' and not containing the critical point -1.*

Proof. Suppose that the critical orbit avoids all non-critical puzzle pieces of depth 1 touching at α'. Recall that these puzzle pieces contain either all rays in Ω_∞ of angles less than 1/2 or all rays in Ω_∞ of angles bigger than 1/2. Thus, all numbers $2^n\theta_0$, $n = 1, 2, \dots$, avoid either $(0, 1/2)$ or $(1/2, 1)$, which contradicts Proposition 5.5. □

Unfortunately, unlike the case of quadratic polynomials, not all the puzzle pieces of depth 1 from Proposition 5.6 are compactly contained in the critical puzzle piece of depth 0. We now need to consider two cases: (1) the post-critical set is disjoint with ω, and (2) the critical orbit enters any neighborhood of ω.

Consider the first case. In this case, choose small disks around ω and a_* that are disjoint from the post-critical set. Add these disks to all puzzle pieces of depth 0 to form *thickened puzzle pieces of depth 0. Thickened puzzle pieces of depth* n are defined as the n-fold pullbacks of the thickened puzzle pieces of depth 0. Clearly, for every point z in the post-critical set and any depth n, there is a unique thickened puzzle piece $\widehat{P^{(n)}}(z)$ containing z (for uniqueness, we use that the small disks around ω and a_* are chosen to be disjoint from the post-critical set). Since the thickened puzzle pieces of depth 1 are compactly contained in thickened puzzle pieces of depth 0, we also have $\widehat{P^{(n)}}(z) \Subset \widehat{P^{(n-1)}}(z)$ for any point z in the post-critical set. It follows that the critical tableau is well defined, and the usual tableau technique of Branner–Hubbard–Yoccoz (see e.g. [12, 7]) applies. The result is that the critical thickened puzzle pieces (and, therefore, critical puzzle pieces) shrink to the critical point -1, provided that the map f is non-renormalizable.

Consider the second case. Thickening puzzle pieces does not help in this case because critical thickened puzzle pieces would not be well defined. Note, however, that the set of angles $2^n\theta_0$ (which are regarded modulo 1) contains 0 in its closure. It follows that this set is dense in $\mathbb{R}/\mathbb{Z}$. In particular, the critical orbit enters all puzzle pieces of depth 1 intersecting Ω_∞. For $r_2 \neq 1/4, 3/4$, there is a puzzle piece of depth 1 that intersects Ω_∞ and is compactly contained in the critical puzzle piece of depth 0. Since the critical orbit enters this puzzle piece, there is a nondegenerate critical annulus. We can now apply the tableau technique.

It remains to consider the case, where r_2 is 1/4 or 3/4, and the set of angles $2^n\theta_0$ is dense in $\mathbb{R}/\mathbb{Z}$ (see also Subsection 5.5 above). There are no nondegenerate critical annuli in this case. Note, however, that a point in the critical puzzle piece $P^{(1)}(-1)$ of depth 1 can only return to this piece under an even iteration of f (because $P^{(1)}(-1)$ is disjoint with the boundary of Ω_∞). Therefore, instead of usual critical annuli, we can consider annuli of the form $P^{(n+2)}(-1) - \overline{P^{(n)}(-1)}$, which we call *double critical annuli*. Double critical annuli exist, because there are puzzle pieces of depth 2 compactly contained in $P^{(0)}(-1)$ (see Picture 7). We can apply the tableau technique to the double critical annuli.

We have proved the following:

Proposition 5.7. *If f is not renormalizable, then the critical puzzle pieces converge to the critical point. Moreover, for any point x not on the boundary of a puzzle piece, the nested sequence of puzzle pieces containing x converges to x.*

The last part is a combination of the tableau technique and the standard Koebe distortion principle (the argument goes exactly as for quadratic polynomials). The boundary condition $-1 \in \partial\Omega_0$ actually implies that

Proposition 5.8. *The map f is non-renormalizable.*

Proof. We use an argument similar to that used in [4] for a family of cubic polynomials (the argument in [4] contains a minor mistake, which can be easily corrected). Suppose that f is renormalizable. Consider the *critical end impression S*, i.e. the intersection of the closures of all critical puzzle pieces. From the construction of the bubble puzzle, it is clear that the intersection of S with the boundary of Ω_0 lies in a single prime end impression of Ω_0, namely, in the impression of angle θ_0. On the other hand, the critical end impression must be periodic, therefore, θ_0 is a rational angle. Consider the landing point of the ray $R_0(\theta_0)$. This point is in the intersection of S with the boundary of Ω_0.

There is a quasi-conformal transformation of a neighborhood of S that maps S to the connected Julia J_0 set of some quadratic polynomial p_0. The intersection $S \cap \overline{\Omega}_0$ corresponds to a connected forward invariant compact subset of J_0. Consider a curve γ in the dynamical plane of p_0 that corresponds to the ray $R_0(\theta_0)$. This curve can be extended to a curve relatively closed in the Fatou set of p_0 and invariant under p_0. It belongs to an open forward invariant subset Ω_0' of $Fatou(p_0)$ corresponding to the set Ω_0. Note that the set Ω_0' is disjoint with all its pullbacks under p_0. However, the pullbacks of γ are everywhere dense in the basin of infinity. A contradiction. $\square$

5.7. **Convergence of cells.** In this section, we prove that all nested sequences of cells converge to singletons (Theorem 4.7). We first need to establish the relationship between puzzle pieces and cells.

Lemma 5.9. *The nested sequence of cells $C^{(n)}(\alpha)$ containing α converges to α, i.e.*

$$\bigcap_{n=1}^{\infty} \overline{C^{(n)}(\alpha)} = \{\alpha\}.$$

Proof. By the proof of Proposition 5.4, the closure of $C^{(n)}(\alpha)$ is disjoint with $\overline{\Omega}$ for large n, therefore, it is compactly contained in C_*. There is a well-defined holomorphic branch $f^{-n} : C_* \to C^{(n)}(\alpha)$, which shrinks all Poincaré distances by a definite factor. It follows that the diameter of $C^{(n)}(\alpha)$ tends to 0 as $n \to \infty$. $\square$

Proposition 5.10. *Consider any point x in the Julia set of f different from α and such that the forward orbit of x is disjoint with $\{-1, \omega\}$. Then there is a cell $C(x)$ that contains x in its closure and lies in a puzzle piece of depth* 0.

Proof. Since x does not coincide with α, it avoids the closure of a cell $C^{(n)}(\alpha)$ containing α (this follows from Lemma 5.9). Let N denote the maximal depth of a basilica component intersecting some special path Γ_i but not lying in the cell $C^{(n)}(\alpha)$. It is not hard to see that the cell $C(x) = C^{(N)}(x)$ of depth N lies in some puzzle piece of depth 0. By definition, x belongs to $\overline{C^{(N)}(x)}$. $\square$

The following statement now follows from the convergence of puzzle pieces.

Proposition 5.11. *Let x be any point in the Julia set of f, whose forward orbit is disjoint with $\{-1, \omega\}$. We have*

$$\bigcap_{n=1}^{\infty} \overline{C^{(n)}(x)} = \{x\}.$$

Note that iterated preimages of ω are the only points in the Julia set that lie on the boundaries of puzzle pieces.

Let x be an iterated preimage of -1. Then, for each depth n, there are two cells $C_0^{(n)}(x)$ and $C_1^{(n)}(x)$ having x on the boundary. We can arrange the indexing so that to have

$$C_0^{(n+1)}(x) \subset C_0^{(n)}(x), \quad C_1^{(n+1)}(x) \subset C_1^{(n)}(x).$$

We will also assume that

$$C_0^{(n)}(-1) \subseteq C_0, \quad C_1^{(n)}(-1) \subseteq C_1.$$

Proposition 5.12. *For any iterated preimage x of the critical point -1, we have*

$$\bigcap_{n=1}^{\infty} \overline{C_0^{(n)}(x)} = \bigcap_{n=1}^{\infty} \overline{C_1^{(n)}(x)} = \{x\}.$$

Proof. It suffices to prove this for $x = -1$. Note that $C_0^{(n)}(-1)$ and $C_1^{(n)}(-1)$ are centrally symmetric with respect to -1. If, say, $\alpha \in C_0$, then C_1 is contained in a single puzzle piece of depth 0, namely, in the critical puzzle piece $P^{(0)}(-1)$. The critical orbit returns to $\overline{C}_1$, and hence to $P^{(0)}(-1)$, infinitely many times. Suppose that $f^{\circ m}(-1) \in \overline{C}_1$. Then, by the pullback argument, $C_0^{(m)}(-1)$ or $C_1^{(m)}(-1)$ is contained in $P^{(m-1)}(-1)$, which is the pullback of $P^{(0)}(-1)$ along the critical orbit. Since m can be made arbitrarily large, the diameters of $C_0^{(n)}(-1)$ and $C_1^{(n)}(-1)$ tend to 0 as $n \to \infty$. □

Proof of Theorem 4.7. Consider a nested sequence of cells $C^{(n)}$. The intersection of all $\overline{C^{(n)}}$ is non-empty. Let x be any point in this intersection. If x is not in the backward orbit of $\{-1, \omega\}$, then the convergence follows from Proposition 5.11. If x is an iterated preimage of ω, then the convergence follows from Proposition 5.3. If x is an iterated preimage of -1, then the convergence follows from Proposition 5.12. □

Note that Theorem A follows from Theorem 4.7, because each cell is connected. It also follows that the boundary of each basilica component is locally connected. In particular, all rays land.

5.8. **Encoding of the Julia set.** In this subsection, we encode all points of the Julia set by binary sequences. Our main tool is Theorem 4.7. Consider a cell C of depth n. The *address of C* is a finite binary sequence $\varepsilon_1 \dots \varepsilon_n$ defined as follows. We set $\varepsilon_k = 0$ or 1 depending on whether $f^{\circ k-1}(C)$ is contained in C_0 or in C_1. We will think of the main cell as having the empty address. For any finite binary sequence $\varepsilon_1 \dots \varepsilon_n$, there is a unique cell $C_{\varepsilon_1 \dots \varepsilon_n}$ with address $\varepsilon_1 \dots \varepsilon_n$. We have $f(C_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_n}) = C_{\varepsilon_2 \dots \varepsilon_n}$.

We can now define a continuous map from all infinite binary sequences to the Julia set of f. Given an infinite binary sequence $\varepsilon_1 \dots \varepsilon_n \dots$, define the point $z_{\varepsilon_1 \dots \varepsilon_n \dots}$ to be the only point in $\bigcap_{n=1}^{\infty} \overline{C}_{\varepsilon_1 \dots \varepsilon_n}$. We have

$$f(z_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_n \dots}) = z_{\varepsilon_2 \dots \varepsilon_n \dots}.$$

The sequence $\varepsilon_1 \dots \varepsilon_n \dots$ is called an *address* of the point $z_{\varepsilon_1\dots\varepsilon_n\dots}$. Note that the same point can have different addresses.

From now on, we will assume that the cells C_0 and C_1 of depth 1 are indexed so that the landing points of all rays $R_\infty(\theta)$ with $\theta < 1/2$ belong to the closure of C_0. Then the landing points of all rays $R_\infty(\theta)$ with $\theta > 1/2$ belong to the closure of C_1. Clearly, this can be arranged.

Proposition 5.13. *The critical point -1 is encoded by exactly two binary sequences, namely,*

$$-1 = z_{0\varepsilon_1^*\dots\varepsilon_n^*\dots} = z_{1\varepsilon_1^*\dots\varepsilon_n^*\dots}, \quad \varepsilon_{2m}^* = \theta_0[m], \quad \varepsilon_{2m+1}^* = 1 - \nu_m(\theta_0),$$

where $\theta_0[m]$ denotes the m-th digit in the binary expression of θ_0, and the function ν_m is that introduced in Subsection 2.1.

Proof. The point -1 belongs to the closures of both C_0 and C_1. However, the remaining address of -1 is well-defined: the m-th digit is 0 if $f^{\circ m-1}(-1)$ belongs to $\overline{C}_0$ and 1 if $f^{\circ m-1}(-1)$ belongs to $\overline{C}_1$. We assumed that -1 is not pre-periodic, thus $f^{\circ m-1}(-1)$ cannot belong to the intersection $\overline{C}_0 \cap \overline{C}_1$, and the m-th digit in the address of -1 is well defined. Denote the m-th digit by ε_m^*.

The point $f^{\circ 2m}(-1)$ is on the boundary of Ω_0. This is the landing point of the ray $R_0(2^m\theta_0)$. It belongs to the closure of C_1 or C_0 depending on whether $\{2^m\theta_0\} < \theta_0$ or $\{2^m\theta_0\} > \theta_0$. Therefore, $\varepsilon_{2m+1} = 1 - \nu_m(\theta_0)$. The point $f^{\circ 2m-1}(-1)$ is on the boundary of Ω_∞. This is the landing point of the ray $R_\infty(2^{m-1}\theta_0)$. It belongs to the closure of C_0 or C_1 depending on whether $\{2^{m-1}\theta_0\} < 1/2$ or $\{2^{m-1}\theta_0\} > 1/2$. Therefore, $\varepsilon_{2m} = \theta_0[m]$. □

Define the following equivalence relation $\sim$ on the set of all infinite binary sequences: $x \sim y$ if and only if one of the following formulas holds:

- $x = 010101\dots$, $y = 101010\dots$,
- $x = w0010101\dots$, $y = w1101010\dots$,
- $x = w0\varepsilon_1^* \dots \varepsilon_n^* \dots$, $y = w1\varepsilon_1^* \dots \varepsilon_n^* \dots$,

for some finite binary word w.

Proposition 5.14. *Let x and y be two infinite binary sequences. We have $z_x = z_y$ if and only if $x \sim y$.*

Proof. In one direction, the proposition is obvious: if x and y are as described, then $z_x = z_y$. Suppose now that $z_x = z_y$. Interchanging x and y if necessary, we can write $x = w0x'$ and $y = w1y'$ for some finite binary word w (possibly empty) and infinite binary sequences x' and y'. We have $z_{0x'} = z_{1y'}$. But $z_{0x'}$ belongs to $\overline{C}_0$, whereas $z_{1y'}$ belongs to $\overline{C}_1$. Note that the sets $\overline{C}_0$ and $\overline{C}_1$ intersect at only three points: ω, -1 and a_*. Consider these three cases separately.

Case 1. Suppose first that $z_{0x'} = z_{1y'} = \omega$. In this case, $x' = 101010\dots$ and $y' = 010101\dots$. Indeed, if a cell lies in C_0 and touches the fixed point ω, then the image of this cell lies in C_1, and vice versa.

Case 2. Suppose that $z_{0x'} = z_{1y'} = a_*$. In this case, it is easy to see that $x' = 010101\dots$ and $y' = 101010\dots$. This follows from the fact that $f(a_*) = \omega$.

Case 3. Finally, suppose that $z_{0x'} = z_{1y'} = -1$. Then $x' = y' = x_0$ by Proposition 5.13. □

Corollary 5.15. *The Julia set of f is homeomorphic to the quotient of the space $\{0,1\}^{\mathbb{N}}$ of all infinite binary sequences (equipped with the product topology) by the equivalence relation $\sim$. Moreover, the canonical projection semi-conjugates the Bernoulli shift with the restriction of f to the Julia set.*

5.9. **Proof of Theorem B.** Consider the two-sided lamination $2L(x_0)$, where x_0 is given in terms of θ_0 by the formula from Theorem B. Let us prove that the Julia set of f is homeomorphic to the quotient of the unit circle by the equivalence relation $\sim_{2L(x_0)}$, and that the map f is conjugate to the map $s_{2L(x_0)}/\sim_{2L(x_0)}$.

We can describe the equivalence relation $\sim_{2L(x_0)}$ in terms of binary digits as follows. Identify each point $e^{2\pi i\theta}$ on the unit circle with the binary expansion of θ, in which each second digit is replaced with its opposite. Under this identification, the map $z \mapsto 1/z^2$ identifies with the Bernoulli shift.

The equivalence relation $\sim_{2L(x_0)}$ is given by the following formulas:

- $101010\cdots \sim 010101\dots$,
- $w001010\cdots \sim w11010\dots$,
- $w0\varepsilon_1^*\dots\varepsilon_n^*\cdots \sim w1\varepsilon_1^*\dots\varepsilon_n^*\dots$.

Note that the first two formulas represent identifications on the unit circle (due to the fact that the same point on the unit circle can correspond to different binary expansions), and only the last formula represents the equivalence defined by the lamination $2L(x_0)$. The digits ε_m^* are the same as in Proposition 5.13 due to Proposition 2.3.

We see that the equivalence relation on binary sequences corresponding to the relation $\sim_{2L(x_0)}$ is identical with that introduced in Subsection 5.8. Thus both $S^1/\sim_{2L(x_0)}$ and the Julia set of the map f are identified with the quotient of the space of infinite binary sequences by the same equivalence relation. It follows that these two sets are homeomorphic. Moreover, both $s_{2L(x_0)}/\sim_{2L(x_0)}$ and f are represented by the Bernoulli shift on binary sequences. Thus the two maps are topologically conjugate.

It is easy to extend the conjugacy $(S^1/\sim_{2L(x_0)}, s_{2L(x_0)}) \to (J, f)$ over the gaps of the lamination $2L(x_0)$. This finishes the proof of Theorem B.

References

[1] L. Ahlfors "Lectures on quasiconformal mappings", Van Nostrand, Princeton, 1966

[2] D. Ahmadi, "Dynamics of certain rational maps of degree two", PhD Thesis, University of Liverpool

[3] A. Douady and J. Hubbard, "Étude dynamique des polynômes complex I& II" Publ. Math. Orsay (1984–85)

[4] D. Faught, “Local connectivity in a family of cubic polynomials”, PhD Thesis, Cornell University, 1992
[5] A. Katok and B. Hasselblatt, “Introduction to the modern theory of dynamical systems” Cambridge University Press, 1995
[6] J. Luo, “Combinatorics and Holomorphic Dynamics: Captures, Matings and Newton’s Method”, PhD Thesis, Cornell University, 1995
[7] M. Lyubich, “Six lectures on real and complex dynamics”, preprint
[8] C. McMullen, “Complex dynamics and renormalization”, Princeton University Press, 1994
[9] C. McMullen, “Automorphisms of rational maps”, In “Holomorphic Functions and Moduli I”, pages 31-60. Springer-Verlag, 1988.
[10] J. Milnor, “Dynamics in One Complex Variable”, Third Edition, Princeton University Press, 2006
[11] J. Milnor, “Geometry and Dynamics of Quadratic Rational Maps” Experimental Math. **2** (1993) 37–83
[12] J. Milnor, “Local connectivity of Julia sets: expository lectures”, in “The Mandelbrot set, Theme and Variations,” LMS Lecture Note Series **274**, Cambr. U. Press (2000), 67–116
[13] R. Perez-Marco, “Fixed points and circle maps”, Acta Math. **179** (1997), 243–294
[14] R. Perez-Marco, “Topology of Julia sets and hedgehogs”, Technical Report **48**, Universite de Paris-Sud, 1994
[15] M. Rees, “Components of degree two hyperbolic rational maps” Invent. Math. **100** (1990), 357–382
[16] M. Rees, “A Fundamental Domain for V_3”, preprint
[17] M. Shub, “Endomorphisms of compact differentiable manifolds”, Amer. J. Math. **91** (1969), 175–199.
[18] D. Sullivan, “Differentiable structures on fractal-like sets, determined by intrinsic scaling functions on dual Cantor sets”, Proc. Sympos. Pure Math. **48** (1988), 15–23.
[19] D. Sullivan, “Conformal dynamical systems”, In “Geometric Dynamics”, Springer-Verlag Lecture Notes **1007** (1983), 725–752.
[20] L. Tan, “Matings of quadratic polynomials”, Erg. Th. and Dyn. Sys. **12** (1992) 589–620
[21] L. Tan and Y. Yin, “Local connectivity of the Julia set for geometrically finite rational maps”, Sci. China Ser. A **39** (1996) 39–47
[22] W. Thurston, “Combinatorics and dynamics of rational maps” Princeton University and IAS Preprint. 1985
[23] D. Tischler, “Blaschke products and expanding maps of the circle”, Proc. Amer. Math. Soc. **128** (1999), No 2, 621–622
[24] B. Wittner, “On the bifurcation loci of rational maps of degree two”, PhD Thesis, Cornell University, 1988
[25] M. Yampolsky and S. Zakeri, “Mating Siegel quadratic polynomials”, J. Amer. Math. Soc. **14**, No. 1, 25–78