EXISTENCE OF POLYGONAL WRAPPINGS IN HYPERBOLIC 3-MANIFOLDS

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Abstract. We will present a short elementary proof of an existence theorem of certain CAT(-1)-surfaces in open hyperbolic 3-manifolds. The main construction lemma in Calegari-Gabai's proof of Marden's Tameness Conjecture can be replaced by an applicable version of our theorem. Finally, we will give a short proof of the conjecture along their ideas.

Very recently, Agol [1] and Calegari-Gabai [5] proved independently that any hyperbolic 3-manifold M with finitely generated fundamental group is homeomorphic to the interior of a compact 3-manifold. This is the affirmative answer to Marden's Tameness Conjecture in [8]. Choi [7] also gives another proof of the conjecture associated with that of Agol's when M has no parabolic cusps.

We are here interested in arguments in [5], where the notion 'shrinkwrapping' was introduced. Shrinkwrappings play an important role in their proof. For the proof of the existence of shrinkwrappings and that of their CAT(-1)-property, Calegari and Gabai used very deep and rarefied arguments, which some of readers including the author may find difficult to approach. This paper is intended to give a rather elementary proof of some part of their proof by using polygonal wrappings instead of shrinkwrappings.

For simplicity, we only consider the case when a hyperbolic 3-manifold has no parabolic cusps, and will prove the following theorem.

Theorem 0.1. *Let N be an orientable hyperbolic 3-manifold without parabolic cusps, Δ a disjoint union of finitely many simple closed geodesics in N, and $f : \Sigma \longrightarrow N$ a 2-incompressible map rel. Δ from a closed orientable surface Σ of genus > 1 to $N \setminus \Delta$. Then, there exists a homotopy $F : \Sigma \times [0, 1] \longrightarrow N$ satisfying the following conditions.*

(i) *$F(x, 0) = f(x)$ for any $x \in \Sigma$.*
(ii) *$F(\Sigma \times [0, 1)) \cap \Delta = \emptyset$.*
(iii) *The map $g : \Sigma \longrightarrow N$ defined by $g(x) = F(x, 1)$ $(x \in \Sigma)$ is a CAT(-1)-polygonal map.*

Here, a continuous map $f : \Sigma \longrightarrow N$ is said to be 2-*incompressible* in N rel. Δ if $f(\Sigma) \cap \Delta = \emptyset$, $f_* : \pi_1(\Sigma) \longrightarrow \pi_1(N \setminus \Delta)$ is injective, and $f|l$ is not freely homotopic in $N \setminus \Delta$ to a (multiplied) meridian of any component of Δ for any simple non-contractible loop l in Σ. See Definition 1.3 for the definition of polygonal maps. We say that a map g satisfying the properties (i)-(iii) as above or its image $g(\Sigma)$ is a CAT(-1)-*polygonal wrapping* of Δ in N homotopic to f. In fact, Theorem 0.1 is a

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special case of Proposition 2.1 which corresponds to the main construction lemma in [5].

In §4, we will give a short proof of Marden's Conjecture along ideas in [5]. Our proof is rather self-contained in the sense that it does not so much invoke published partial solutions to the conjecture.

1. Completion of certain hyperbolic manifolds

For a closed subset A in a metric space (X, d), the r-neighborhood of A in X, $\{y \in X;\, d(y, A) \le r\}$, is denoted by $\mathcal{N}_r(A)$ (or more strictly by $\mathcal{N}_r(A, X)$). In the case when A is a single point set $\{x\}$, we also set $\mathcal{N}_r(\{x\}) = \mathcal{B}_r(x)$. The link of x in X with radius r, $\{y \in X;\, d(y, x) = r\}$, is denoted by $\mathcal{S}_r(x)$.

Let U be a simply connected incomplete hyperbolic 3-manifold with the metric completion $\overline{U}$ such that each component l of $L = \overline{U} \setminus U$ is a line and there exists a constant $c > 0$ with $\mathrm{dist}(x, y) \ge 3c$ for any points x, y contained in mutually distinct components of L. Moreover, we suppose that, for any component l of L, there exists an infinite cyclic branched covering $p_l : \mathcal{N}_c(l, \overline{U}) \longrightarrow \mathcal{N}_c(j, \mathbf{H}^3)$ branched over a geodesic line j of $\mathbf{H}^3$ such that the restriction $p_l|\mathcal{N}_c(l, \overline{U}) \setminus l$ is a locally isometric covering. From the definition, we know that $\mathcal{N}_c(l, \overline{U})$ is homeomorphic to the quotient space of $\mathbf{R}^2 \times [0, 1]$ by the identification map $a : \mathbf{R}^2 \times \{0\} \longrightarrow \mathbf{R}$ defined by $a(x, y, 0) = x$. In particular, $\partial\mathcal{N}_c(l)$ is homeomorphic to a disjoint union of a single plane $\mathbf{R}^2$ and a line $\mathbf{R}$. Any geodesic segment in $\overline{U}$ is a broken line consisting of finitely many hyperbolic segments. Any vertex of the broken line other than its end points lies in L.

The hyperbolic metric on $\mathbf{H}^2$ is represented in polar coordinate as $dr^2 + \sinh^2 r\, d\theta^2$, where $r \ge 0$ is the distance to a fixed point $x \in \mathbf{H}^2$ and $\theta \in [0, 2\pi]$ is the length parameter on the unit circle in $\mathbf{H}^2$ centered at x. From this, we know that the hyperbolic metric on $\mathcal{N}_c(j, \mathbf{H}^3)$ is represented as

$$ds^2 = dr^2 + \sinh^2 r\, d\theta^2 + \cosh^2 r\, du^2,$$

where du^2 is the hyperbolic metric on j.

Fix a smooth function $\tau : (0, c] \longrightarrow \mathbf{R}$ such that (i) $\tau' \le 0$, (ii) $\tau(t) = 1$ for any t with $c/2 \le t \le c$, and (iii) $\displaystyle\int_0^c \tau(t)\, dt = \infty$. Consider the Riemannian metric g_n on $\mathcal{N}_c(j, \mathbf{H}^3) \setminus j$ $(n \in \mathbf{N})$ such that the restriction $g_n|\mathcal{N}_n(j) \setminus \mathcal{N}_{c/n}(j)$ is the standard hyperbolic metric, and $g_n|\mathcal{N}_{c/n}(j) \setminus j$ is represented as

$$ds_n^2 = \tau_n(r)^2 dr^2 + \sinh^2 r\, d\theta^2 + \cosh^2 r\, du^2,$$

where $\tau_n : (0, c/n] \longrightarrow \mathbf{R}$ is the smooth function defined by $\tau_n(t) = \tau(nt)$. The condition (iii) on τ implies that the metric is complete. For the coordinate $(r, \theta, u) = (x_1, x_2, x_3)$, the Riemannian curvature tensors of g_n in $\mathcal{N}_{c/n}(j) \setminus j$ are given by

$$\begin{aligned}
R_{1212} &= -\sinh^2 r + \frac{\sinh r \cosh r\, \tau_n'(r)}{\tau_n(r)}, \\
R_{1313} &= -\cosh^2 r + \frac{\sinh r \cosh r\, \tau_n'(r)}{\tau_n(r)}, \\
R_{2323} &= -\frac{\sinh^2 r \cosh^2 r}{\tau_n(r)^2}.
\end{aligned}$$

Since $\tau_n'(r) \le 0$ and $\tau_n(r) > 0$ for $r \in (0, c/n]$, it follows that the metric g_n has negative curvature.

Consider the Riemannian metric h_n on U replacing the hyperbolic metric on $\mathcal{N}_c(l, \overline{U}) \setminus l$ by $p_l^*(g_n)$ for any component l of L. The original metric on U is the limit of the complete negatively curved metrics h_n. Since any geodesic segment σ in $\overline{U}$ is a uniformly convergent limit of geodesic segments σ_n in (U, h_n) with $\lim_{n\to\infty} \mathrm{length}_{h_n}(\sigma_n) = \mathrm{length}_{\overline{U}}(\sigma)$, $\overline{U}$ is a CAT(0)-space, see Bridson-Haefliger [3, Chapter II.1] for the definition and properties of such spaces. From this, we know that a geodesic segment σ in $\overline{U}$ connecting given two points is uniquely determined. Moreover, σ moves continuously along the continuous deformation of its end points.

Consider an interior vertex x of a segment σ, and let σ_1, σ_2 be short hyperbolic segments in σ with $\sigma_1 \cap \sigma_2 = \{x\}$. Let x_i $(i = 1, 2)$ be the point in σ_i with $\mathrm{dist}(x_1, l) = \mathrm{dist}(x_2, l) = s > 0$, where l is the component of L containing x, see Fig. 1.1 (a). There exist totally geodesic half planes P_i in $\overline{U}$ with $\sigma_i \subset P_i$ and

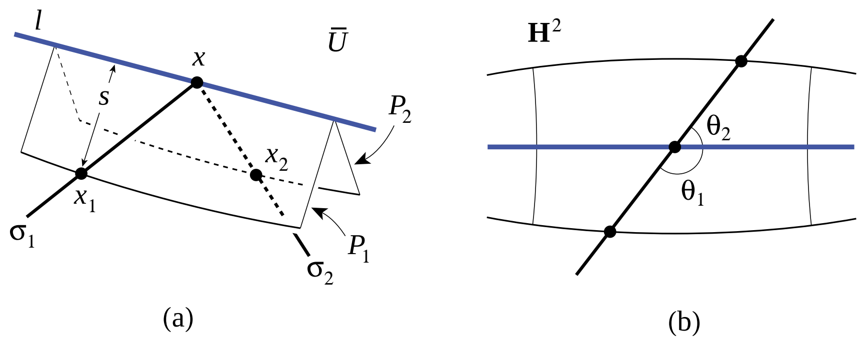

FIGURE 1.1

$\partial P_i = l$. Since the subsegment τ of σ with $\partial\tau = \{x_1, x_2\}$ is the shortest arc in $\overline{U}$ connecting x_1 with x_2, we have

$$\theta_1 + \theta_2 = \pi, \tag{1.1}$$

where θ_i is the angle made by σ_i and a fixed ray in l emanating from x. This fact is easily seen by considering the developing of $P_1 \cup P_2$ on $\mathbf{H}^2$, see Fig. 1.1 (b).

For any d with $0 < d \le c$, $\mathcal{B}_d(x, \overline{U})$ is homeomorphic to the subset of $\mathbf{R}^3$;

$$\{(u, v, w) \in \mathbf{R}^3;\ u^2 + v^2 + w^2 \le 1, w > 0\} \cup \{(u, 0, 0) \in \mathbf{R}^3;\ -1 \le u \le 1\}.$$

In particular, $\mathcal{B}_d(x, \overline{U})$ is simply connected. The image $p_l(\mathcal{B}_d(x, \overline{U}))$ coincides with the hyperbolic ball $\mathcal{B}_d(\hat{x}, \mathbf{H}^3)$, where $\hat{x} = p_l(x)$. Rescaling the metric on the boundary $S = \mathcal{S}_d(\hat{x}, \mathbf{H}^3)$ of the ball, we have the spherical metric ν on S isometric to the unit sphere in the Euclidean 3-space. Consider the metric on $\tilde{S} = \mathcal{S}_d(x, \overline{U})$, still denoted by ν, so that the infinite cyclic branched covering $p_l|\tilde{S} : \tilde{S} \longrightarrow S$ is locally pathwise isometric. Here, $p_l|\tilde{S}$ being *locally pathwise isometric* means that $\mathrm{length}_\nu(\alpha) = \mathrm{length}_\nu(p_l(\alpha))$ for any rectifiable arc α in $\tilde{S}$. One can take $d > 0$ so that $\sigma' = \sigma \cap \mathcal{B}_d(x)$ is an embedded arc in $\mathcal{B}_d(x)$ with $\partial\sigma' \subset \tilde{S}$. Let γ be any rectifiable arc in $\tilde{S}$ with $\partial\gamma = \partial\sigma'$. Since $\mathcal{B}_d(x)$ is simply connected, γ is homotopic rel. $\partial\gamma$ to σ' in $\mathcal{B}_d(x)$. Then, the following lemma is proved immediately from the equality (1.1) and by checking the situation of $\hat{\gamma} = p_l(\gamma)$ in S, see Fig. 1.2.

Lemma 1.1. $\mathrm{length}_\nu(\gamma) \geq \pi$.

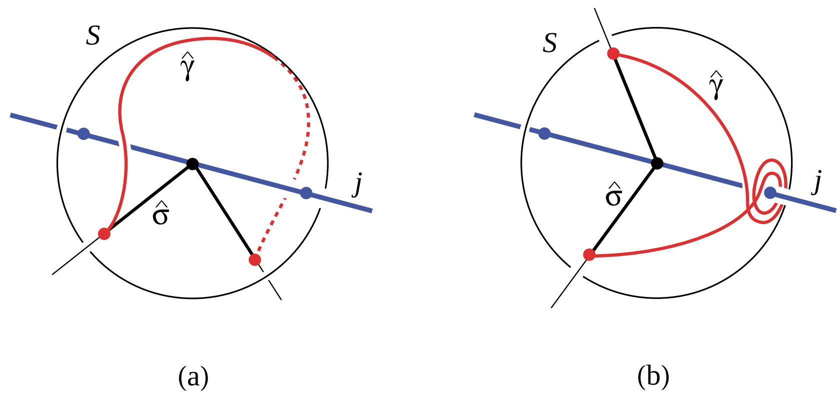

FIGURE 1.2. $j = p_l(l)$, $\hat{\sigma} = p_l(\sigma')$. (b) is the case when $\hat{\gamma}$ winds around j more than once, but $\mathrm{length}_\nu(\gamma) = \mathrm{length}_\nu(\hat{\gamma})$ does not exceed π very much.

Let Z be an incomplete hyperbolic 3-manifold such that the total space of the universal covering $q : U \longrightarrow Z$ has the induced metric as above. We suppose moreover that, for the metric completion $\overline{Z}$, each component l of $\overline{Z} \setminus Z$ is either a geodesic line or a geodesic loop. That is, any component of $\overline{Z} \setminus Z$ is not a one-point set. Then, $q : U \longrightarrow Z$ is extended to a locally pathwise isometric map $\overline{q} : \overline{U} \longrightarrow \overline{Z}$. Note that the frontier $\{x \in \overline{Z};\, \mathrm{dist}(x, l) = c\}$ of $\mathcal{N}_c(l, \overline{Z})$ in $\overline{Z}$ is homeomorphic to either $\mathbf{R}^2$ or an open annulus or a torus.

Remark 1.2. Even in the case when $\mathcal{N}_c(l, \overline{Z})$ is homeomorphic to a solid torus, we always suppose that *homotopies in* $\overline{Z}$ starting from a continuous map $f : A \longrightarrow Z$ never cross l (possibly they touch l), where A is a manifold of dimension less than three. In other words, we only consider homotopies $F : A \times [0, 1] \longrightarrow \overline{Z}$ which can be covered by a map $\tilde{F} : \tilde{A} \times [0, 1] \longrightarrow \overline{U}$, where $\tilde{A}$ is the universal covering space of A.

Definition 1.3. Let $f : \Sigma \longrightarrow \overline{Z}$ be a continuous map from a closed orientable surface Σ. Suppose that Σ admits a cell decomposition K consisting of finitely many polygonal 2-cells. We say that f is a *polygonal map* with respect to K if (i) $f^{-1}(\overline{Z} \setminus Z)$ is a union of some vertices and edges in K, (ii) for each edge of e of K, $f(e)$ is a hyperbolic segment, and (iii) for each 2-cell F of K, $f(F)$ is a totally geodesic hyperbolic polygon. Then, the induced metric on Σ is a hyperbolic metric with a cone-type singularity set consisting of vertices of K. Such a map f is called a CAT(-1)-polygonal map if the cone-angle of Σ at any singular point is not less than 2π.

2. Applicable version of Theorem 0.1

Throughout this section, we assume that any hyperbolic 3-manifolds and surfaces are orientable.

Let N be a complete hyperbolic 3-manifold N without parabolic cusps, and W a 3-dimensional compact connected submanifold of N. Consider a link Δ in $\mathrm{Int}W$

consisting of finitely many simple closed geodesics in N. Let $p : X \longrightarrow W$ be the covering of W associated to a finitely generated subgroup of $\pi_1(W)$. Here, we put the following assumptions, which correspond to those in the main construction lemma, Lemma 2.1 in [5].

(i) ∂W is incompressible in $N \setminus \Delta$.
(ii) There exists a union $\hat{\Delta}$ of components of $p^{-1}(\Delta)$ such that the restriction $p|\hat{\Delta} : \hat{\Delta} \longrightarrow \Delta$ is a homeomorphism.
(iii) There exists a continuous map $f : \Sigma \longrightarrow X$ from a closed surface Σ of genus $m > 1$ which is 2-incompressible in X rel. $\hat{\Delta}$.

Set $X^\circ = X \setminus p^{-1}(\Delta)$. The fundamental group $\pi_1(X^\circ)$ may be infinitely generated. By (i), the restriction $p^\circ = p|X^\circ : X^\circ \longrightarrow W \setminus \Delta \subset N \setminus \Delta$ is π_1-injective. Thus, we may assume that X° is a subset of the total space of the covering $q : Y^\circ \longrightarrow N \setminus \Delta$ associated to the subgroup $p^\circ_*(\pi_1(X^\circ))$ of $\pi_1(N \setminus \Delta)$ and the inclusion $i : X^\circ \longrightarrow Y^\circ$ is a homotopy equivalence. Since $\partial X^\circ = \partial X = p^{-1}(\partial W)$ is a deformation retract of $Y^\circ \setminus \mathrm{Int} X^\circ$, the condition (iii) implies that $f : \Sigma \longrightarrow Y$ is 2-incompressible in Y rel. $\hat{\Delta}$, where $Y = X \cup (Y^\circ \setminus \mathrm{Int} X^\circ) = X \cup Y^\circ$. The complement $Z = Y \setminus \hat{\Delta}$ has the induced incomplete metric as was studied in §1. Let $\overline{Z}$ be the metric completion of Z.

With the notation and assumptions as above, we will prove the following proposition.

Proposition 2.1. *There exists a homotopy $F : \Sigma \times [0, 1] \longrightarrow \overline{Z}$ which never crosses $\hat{\Delta}$ and connects f with a* CAT(-1)*-polygonal wrapping $g : \Sigma \longrightarrow \overline{Z}$ of $\hat{\Delta}$.*

Proof. Let $c_1, \dots, c_{3m-3}$ be mutually disjoint simple loops in Σ which define a pants decomposition of Σ. Consider a cell decomposition K of Σ consisting of triangular 2-cells and such that each vertex of K is contained in $c_1 \cup \cdots \cup c_{3m-3}$. If necessary deforming f by homotopy in the sense of Remark 1.2, we may assume that each $f(c_i)$ is a closed geodesic in $\overline{Z}$, and $f(e)$ is a geodesic segment in $\overline{Z}$ for any edge e of K not contained in $c_1 \cup \cdots \cup c_{3m-3}$. In fact, $f(c_i)$ is the image of an axis of a hyperbolic transformation on $\overline{U}$, see for example [3, Theorem 6.8 (1)]. For any 2-cell F of K, take a vertex v_0 and the opposite edge e_0. Then, $f|F$ can be homotoped rel. ∂F to a map $g|F$ such that $g(F)$ is a ruled triangle consisting of all geodesic segments connecting $f(v_0)$ with points of $f(e_0)$. These $g|F$ define a map $g : \Sigma \longrightarrow \overline{Z}$ homotopic to f. From our construction of g, there exists a subdivision K' of K with respect to which g is a polygonal map. Moreover, for any vertex v of K', there exists an arc α in Σ with $\mathrm{Int}\alpha \ni v$ and such that $g(\alpha)$ is a geodesic segment in $\overline{Z}$. If $g(v)$ is not an element of $\overline{Z} \setminus Z$, then it is easily seen that the cone-angle of Σ at v is not less than 2π. So, we may assume that $g(v)$ is contained in a component l of $\overline{Z} \setminus Z$. For a sufficiently small $d > 0$, α divides the circle $\mathcal{S}_d(v, \Sigma)$ into two arcs γ_1, γ_2. By Lemma 1.1, the ν-length of $g(\gamma_i)$ $(i = 1, 2)$ in $\mathcal{S}_d(g(v), \overline{Z})$ is not less than π. Thus, the cone-angle of Σ at v is not less than 2π. This shows that g is a CAT(-1)-polygonal wrapping of $\hat{\Delta}$ in $\overline{Z}$. □

Note that Theorem 0.1 is proved quite similarly to Proposition 2.1 by considering (N, Δ) instead of $(\overline{Z}, \hat{\Delta})$.

3. Compact cores and end reductions

A 3-manifold X is *topologically tame* if there exists an embedding $f : X \longrightarrow Y$ into a compact manifold Y with $f(X) \supset \mathrm{Int}Y$. Throughout this section, we suppose that M is an orientable, open, irreducible and connected 3-manifold with finitely generated fundamental group. An end $\mathcal{E}$ of M is said to be *topologically tame* if there exits a closed neighborhood of $\mathcal{E}$ in M homeomorphic to $S \times [0, \infty)$ for some closed connected surface S. It is easily seen that the open 3-manifold M is topologically tame if and only if each end of M is so.

Scott [10] proved that M contains a 3-dimensional submanifold C, called a *compact core* of M, such that the inclusion $i : C \longrightarrow M$ is a homotopy equivalence. Let S be the component of ∂C facing an end $\mathcal{E}$ of M, and $p : \tilde{M} \longrightarrow M$ the covering associated with the image of $\pi_1(S)$ in $\pi_1(M)$. There exists a compact core $\tilde{C}$ of $\tilde{M}$ such that $\partial\tilde{C}$ has a component $\tilde{S}$ mapped onto S homeomorphically by p. The manifold $\tilde{C}$ is a *compression body*, that is, it is homeomorphic to $E \cup h_1 \cup \cdots \cup h_m$ where E is either a 3-ball or $F \times [0, 1]$ for some closed surface F consisting of non-spherical components and h_i's are 1-handles attached to a one-side of E. In particular, when E is a 3-ball, the compression body $\tilde{C}$ is a handlebody. Note that the end $\tilde{\mathcal{E}}$ of $\tilde{M}$ faced by $\tilde{S}$ is topologically tame if and only if $\mathcal{E}$ is so.

Let $\Delta = \delta_1 \cup \cdots \cup \delta_{i_0}$ be a i_0-component link in the compression body $\tilde{C}$ such that $[\delta_k]$ $(k = 1, \cdots, i_0 - 1)$ form a generator system for $H_1(\tilde{C}, \mathbf{Z})$ and $[\delta_{i_0}] = [\delta_1] + \cdots + [\delta_{i_0-1}]$. An advantage of considering compression bodies is that any non-trivial free decomposition of $\pi_1(\tilde{C})$ induces the non-trivial decomposition of $H_1(\tilde{C}; \mathbf{Z})$. In particular, this implies that the link Δ is *algebraically disk-busting*, that is, for any non-trivial free decomposition $A * B$ of $\pi_1(\tilde{C})$, there exists a component δ_k of Δ such that the element of $\pi_1(\tilde{C})$ represented by δ_k is neither conjugate into A nor B.

Some results in Myers [9] concerning end reductions play an important role in the proof of Theorem 4.1. The paper is useful also as an expository article on end reductions. A compact, connected, 3-dimensional submanifold R of M is *regular* if $M \setminus R$ is irreducible and the closure of any component of $M \setminus R$ in M is not compact. Let Δ be a link in M each component of which is non-contractible in M. An open submanifold V of M containing Δ is called an *end reduction* of M at Δ if it satisfies the following condition.

(i) No component of $M \setminus V$ is compact.
(ii) There exists a sequence $\{R_n\}$ of regular submanifold of M with $\Delta \subset R_1$, $R_n \subset \mathrm{Int}R_{n+1}$, $V = \bigcup_n R_n$ and such that ∂R_n is incompressible in $M \setminus \Delta$.
(iii) V satisfies the *engulfing property* at Δ, that is, for any regular submanifold N of M with $\Delta \subset \mathrm{Int}N$ such that ∂N is incompressible in $M \setminus \Delta$, V is ambient isotopic rel. Δ to a manifold containing N.

We refer to Brin-Thckstun [4] for the existence and uniqueness up to isotopy of end reductions. According to Myers [9, Theorem 9.2], if the link Δ is algebraically disk-busting, then an end reduction V of M at Δ is connected and the homomorphism $i_* : \pi_1(V) \longrightarrow \pi_1(M)$ induced from the inclusion is isomorphic.

4. Proof of Marden's Conjecture

Our proof of Marden's Conjecture is based on that of Calegari-Gabai [5], but the importance of 'disk-busting' is suggested by Agol [1]. We only consider hyperbolic

3-manifolds without parabolic cusps just for simplicity. It is not hard to modify our argument applicable to the case when manifolds have parabolic cusps.

Theorem 4.1 (Marden's Tameness Conjecture)**.** *Let N be an orientable hyperbolic 3-manifold without parabolic cusps. If $\pi_1(N)$ is finitely generated, then N is topologically tame.*

Proof. It suffices to show that each end $\mathcal{E}$ of N is topologically tame. As was seen in §3, we may assume that a compact core C of N is a compression body. Let S be the component of ∂C facing $\mathcal{E}$. If $\mathcal{E}$ is *geometrically finite*, that is, C is locally convex in S, then it is well known that $\mathcal{E}$ is topologically tame, for example see Marden [8]. So, we may assume that $\mathcal{E}$ is not geometrically finite. Then, Bonahon [2] shows that there exists a sequence $\{\delta_i\}$ of closed geodesics in M exiting $\mathcal{E}$. If necessary adding finitely many closed geodesics to $\{\delta_i\}$, one can suppose that $\Delta_i = \delta_1 \cup \cdots \cup \delta_i$ is algebraically disk-busting if i is not less than some fixed integer $i_0 > 0$. If necessarily slightly deforming the hyperbolic metric in a small neighborhood of $\bigcup_i \delta_i$ in N, we may assume that the closed geodesics δ_i are simple and mutually disjoint, i.e. each Δ_i is a link in N. In fact, the resulting metric is no longer hyperbolic but pinched negatively curved. However, all the results concerting hyperbolic manifolds which we need, e.g. Proposition 2.1 in §2, still hold under this metric.

For any $i \geq i_0$, let V_i be an end reduction of N at Δ_i. By [9], the homomorphism induced from the inclusion $V_i \longrightarrow N$ is isomorphic. It follows that V_i contains a compact core C_i of N, and the free homotopy in N between δ_k $(k = 1, \ldots, i)$ and a loop in C_i is realized in V_i. By the property (i) of the end reduction V_i, there exists a regular submanifold W_i of V_i containing both C_i and the traces of these free homotopies and such that ∂W_i is incompressible in $N \setminus \Delta_i$. If necessary deforming C_i by isotopy in ∂W_i, we may assume that $\partial C_i \setminus S_i$ is contained in ∂W_i, where S_i is the component of ∂C_i facing $\mathcal{E}$. Since the inclusion $C_i \longrightarrow W_i \longrightarrow N$ is π_1-isomorphic, $\pi_1(C_i)$ can be regarded as a subgroup of $\pi_1(W_i)$. Consider the covering $p_i : X_i \longrightarrow W_i$ associated to $\pi_1(C_i) \subset \pi_1(W_i)$. Let $\hat{\delta}_k$ be a component of $p_i^{-1}(\delta_k)$ such that $p_i|\hat{\delta}_k : \hat{\delta}_k \longrightarrow \delta_k$ is homeomorphic, and set $\hat{\Delta}_i = \hat{\delta}_1 \cup \cdots \cup \hat{\delta}_i$.

Here, we will show that X_i is topologically tame. Let $\mathcal{T}_i = T_1 \cup \cdots \cup T_m$ be a maximal union of mutually disjoint and non-parallel incompressible tori in $\mathrm{Int}W_i$. Since W_i is regular and N is atoroidal and irreducible, each T_j bounds a compact 3-manifold A_j in W_i homeomorphic to the exterior of a non-trivial knot in S^3 and contained in a 3-ball in N. Either any two A_j are mutually disjoint or one of them contains the other. Set $\mathcal{A} = A_1 \cup \cdots \cup A_m$. Note that, for any component $\hat{A}$ of $p_i^{-1}(\mathcal{A})$, the image $\mathrm{inc}_i \circ p_i(\hat{A})$ is contained in a 3-ball in N, where $\mathrm{inc}_i : W_i \longrightarrow N$ is the inclusion. Since $\mathrm{inc}_i \circ p_i : X_i \longrightarrow N$ is π_1-isomorphic, it follows that $\hat{A}$ is simply connected. Then, by Waldhausen [14, Theorem 8.1], $\hat{A}$ is topologically tame. Since each component of $\partial\hat{A}$ is simply connected, $p_i^{-1}(\partial\mathcal{A})$ induces a free decomposition of $\pi_1(X_i)$. The classical Grushko Theorem implies that the fundamental group of any component $\hat{B}$ of $p_i^{-1}(W_i \setminus \mathrm{Int}\mathcal{A})$ is finitely generated. Since $W_i \setminus \mathrm{Int}\mathcal{A}$ is an atoroidal Haken manifold such that one of the boundary components has genus > 1, by Canary [6, Proposition 3.2], $\mathrm{Int}\hat{B}$ is topologically tame. In the present case, it is not hard to show that $\hat{B}$ is also topologically tame, see Soma [12] for general case. Finally, Simon's Combination Theorem [11, Theorem 3.1] implies that X_i is topologically tame.

Let $\overline{X}_i$ be the manifold completion of X_i, and $\overline{S}_i$ the component of $\partial\overline{X}_i$ facing a closed surface in IntX_i mapped onto S_i by p_i. Consider a closed surface $\hat{S}_i$ in IntX_i obtained by a small isotopy of $\overline{S}_i$ in $\overline{X}_i$. We show that $\hat{S}_i$ is 2-incompressible in X_i rel. $\hat{\Delta}_i$. If not, there would exist a compressing disk D for $\hat{S}_i$ such that the intersection $D \cap \hat{\Delta}_i$ consists of at most one point. It follows that $\hat{\Delta}_i$ is not algebraically disk-busting in X_i. Since $(\mathrm{inc}_i \circ p_i)_* : \pi_1(X_i) \longrightarrow \pi_1(N)$ is isomorphic and $\mathrm{inc}_i \circ p_i(\hat{\Delta}_i) = \Delta_i$, the link Δ_i would not be algebraically disk-busting in N, a contradiction.

For any $i \geq i_0$, let $\overline{q}_i : \overline{Z}_i \longrightarrow N$ be the locally pathwise isometric map extending the covering $q_i : Y_i^\circ \longrightarrow N \setminus \Delta_i$ given in §2 which satisfies $q_i = p_i$ on $X_i \setminus p_i^{-1}(\hat{\Delta}_i)$. Note that $\overline{Z}_i$ is the metric completion of $Z_i = X_i \cup Y_i^\circ \setminus \hat{\Delta}_i$. By Proposition 2.1, $\hat{S}_i$ is homotopic in $\overline{Z}_i$ to a CAT(-1)-polygonal wrapping $\hat{\Sigma}_i$ without crossing $\hat{\Delta}_i$. The image $\Sigma_i = \overline{q}_i(\hat{\Sigma}_i)$ is also a CAT(-1)-surface homotopic in N to S_i.

Let $\hat{\alpha}_i$ be a ray in $\overline{Z}_i$ emanating from $\hat{\delta}_i$ which covers a proper ray α_i in N such that each α_i meets the component of ∂W_i facing $\mathcal{E}$ transversely in a single point and the sequence $\{\alpha_i\}$ exits $\mathcal{E}$. Since the algebraic intersection number of $\hat{\alpha}_i$ with $\hat{S}_i$ is one, $\hat{\Sigma}_i \cap \hat{\alpha}_i$ and hence $\Sigma_i \cap \alpha_i$ are not empty. By using this fact together with Bounded Diameter Lemma [5, Lemma 1.4] for CAT(-1)-surfaces, it is not hard to show that $\{\Sigma_i\}$ exits $\mathcal{E}$. Under the present situation, the tameness of $\mathcal{E}$ is proved by standard arguments in hyperbolic geometry, for example see [2, 6, 13] or Tameness Criteria in [5, §6]. □

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