Extra Knot Examples.

Example LRCLRRC, $p = 1/3$.

For this example, we can compute directly and find that $\zeta \mid U$ is a power series in ρ, equivalently, $\pi \mid U$ is degree one and U is smooth. We now do the calculation, which will give some idea of how the calculation will work in general. In this example, 1 is to be periodic of period 7. General theory tells us that ζ has a power series expansion in $\rho^{1/N}$ for some N. We have also seen that $\zeta = 1 + a\rho + o(\rho)$ for some $a \neq 0$. Now from (E0) we have

$$h^2_{\zeta,\rho}(1) = \zeta^2(1 - \tfrac{1}{2}\rho + \frac{5}{8}\rho^2).$$

In order to obtain $h^3_{\zeta,\rho}(1) = 1 + O(\rho)$ (as we must do) we need $h^2_{\zeta,\rho}(1) = 1 + \frac{1}{2}\rho - \frac{1}{8}\rho^2 + O(\rho^3)$, and hence we need $a = 1$, $\zeta = -1(1 + \frac{1}{2}\rho + O(\rho^2))$. Since we need $h^6_{\zeta,\rho}(1) = 1 + O(\rho)$, we also need $h^5_{\zeta,\rho}(1) = 1 + \frac{1}{2}\rho - \frac{1}{8}\rho^2 + O(\rho^3)$. We therefore need $h^3_{\zeta,\rho}(1) = 1 + o(\rho)$.

So now take $1 + v = h^3_{\zeta,\rho}(1)$. Then since $h'_{\zeta,\rho}(-1 + O(\rho)) = -1 + O(\rho)$, from (E1),

$$h^5_{\zeta,\rho}(1) = h^2_{\zeta,\rho}(1) - 2v^2/\rho + O(v^2) + O(v^3/\rho^2).$$

Since we need $h^5_{\zeta,\rho}(\rho) = h^2_{\zeta,\rho}(1) + c\rho^3 + o(\rho^3)$ for some $c \neq 0$, we need $v = d\rho^2 + o(\rho^2)$ for some $d \neq 0$.

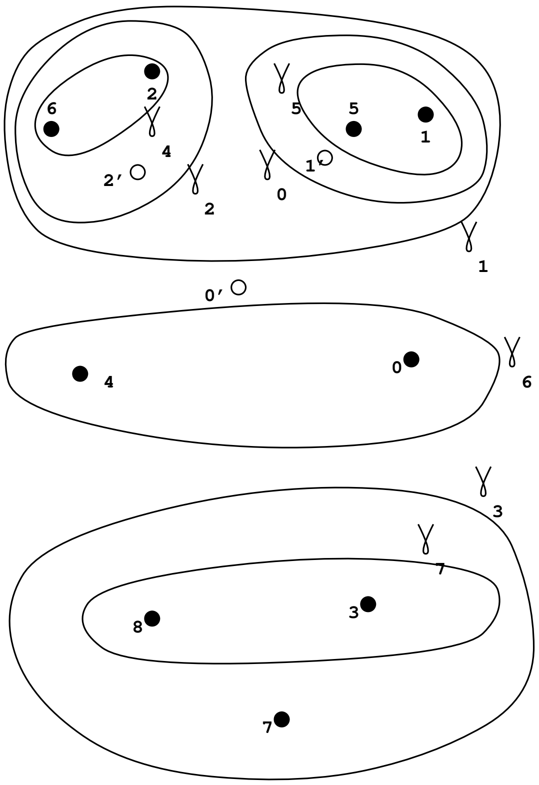
Now we need to show that ζ is a power series in ρ. Suppose for contradiction that ζ is a power series expansion in $\rho^{1/N}$ for some $N \geq 1$ that the lowest noninteger power of ρ in ζ is a term $\rho^{L/M}$ for some coprime L and M with $M \geq 2$. We know that $L/M > 2$ since $h^2_{\zeta,\rho}(1) = 1 + \frac{1}{2}\rho - \frac{1}{8}\rho^2 + o(\rho^2)$. Then the lowest noninteger power in $h^2_{\zeta,\rho}(1)$ is $\rho^{L/M}$. But then the lowest noninteger power in $h^3_{\zeta,\rho}(1)$ is $\rho^{L/M-2}$. Then writing $h^3_{\zeta,\rho}(1) = 1 + v$, since $v = d\rho^2 + o(\rho^2)$ for $d \neq 0$, the lowest noninteger power in v^2 is $\rho^{L/M}$. Then the lowest noninteger power in v^2/ρ is $\rho^{L/M-1}$. This is also the lowest noninteger power in $h^4_{\zeta,\rho}(1) = -1 + O(\rho)$, since the lowest noninteger power in v^2/ρ cannot cancel with that in ζ, and in $h^5_{\zeta,\rho}(1) = 1 + \frac{1}{2}\rho - \frac{1}{8}\rho^2 + O(\rho^3)$. Then by (E2) the lowest noninteger power in $h^6_{\zeta,\rho}(1)$ is $\rho^{L/M-3}$, and in $h^7_{\zeta,\rho}(1)$ is $\rho^{L/M-5}$. Then we obtain a contradiction, because $h^7_{\zeta,\rho}(1) = 1$, and thus there are no noninteger powers in the expansion of $h^7_{\zeta,\rho}(1)$. So there are no noninteger powers in the expansion of $h^2_{\zeta,\rho}(1)$ either, and hence none in the expansion of ζ.

Example LRCLRCLRRC, $p = 1/3$.

It is possible to show that ζ has a power series expansion in powers of $\rho^{1/2}$ by a similar method to that employed for the branch associated to $s_{1/3} \amalg s_{55/127}$ (which was shown to be smooth). So suppose there is lowest t such that the expansion of ζ contains a ρ^t term, $t \notin \mathbf{Z}/2$. Then ρ^{t-2} is in the expansion of $h^3_{\zeta,\rho}(1)$. So $t > 5/2$. Using (E1), the lowest such term in $h^4_{\zeta,\rho}(1)$ is a term $(\rho^{t-2})\rho^{5/2-1} = \rho^{t-1/2}$. This cannot cancel with any term in the ζ expansion. The lowest such term in $h^6_{\zeta,\rho}(1)$ is $\rho^{t-5/2}$ with $t - 3/2 > 2$. Then the lowest term in $h^7_{\zeta,\rho}(1)$ is $(\rho^{t-5/2})\rho^{2-1} = \rho^{t-3/2}$. Then the lowest term in $h^{10}_{\zeta,\rho}(1) = 1$ is $\rho^{t-7/2}$, again giving a contradiction.

Examples where $m = 0$ even though A has noninteger entries.

We now show that the most obvious generalisation of f in the example above to the case when p has period ≥ 3 has $m = 0$, even though the matrix for $f^* : \mathbf{R}^\Gamma \to \mathbf{R}^\Gamma$ in this new example will have noninteger entries. We describe our new branched covering $s_{1/7} \amalg s_r$ by describing s_r. In fact, our description of s_r will be nonunique, and given in terms of the (L, R, C) sequence as before. The sequenceis $LLRCLLRRC$.



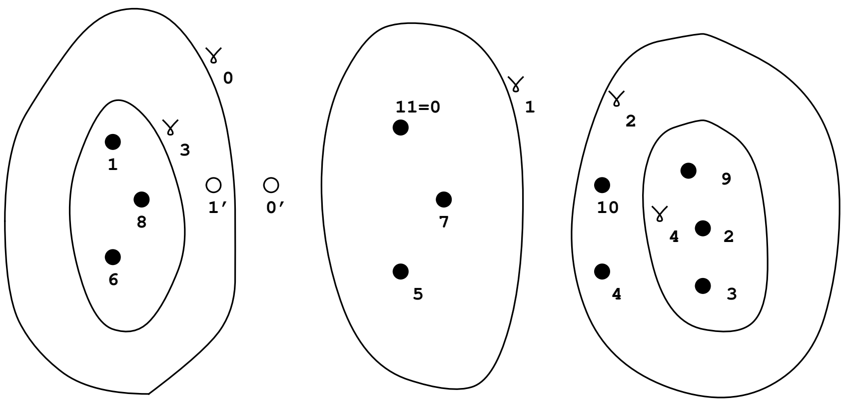
LLRCLLRRC, $p = 1/7$.

There are two choices of $\mu_q > \mu_{6/7}$ for which $s_q(0)$ has this sequence, which might suggest that $m = 1$. However, if we look at the matrix we find the following. Fix such a q and consider $f = s_{1/7} \amalg s_q$. Let $\gamma_0 = \mu_{1/7} \cup \mu_{6/7}^{-1}$. Let γ_1 be the periodic component of $f^{-1}(\gamma_0)$ and γ_2 the periodic component of $f^{-1}(\gamma_1)$. Let γ_3 be the nonperiodic component of $f^{-1}(\gamma_0)$. The nonperiodic component of $f^{-1}(\gamma_2)$ is trivial. Hence γ_3 is also homotopic to the nonperiodic component of $f^{-1}(\gamma_1)$. The only nonperipheral component of $f^{-1}(\gamma_3)$ is that separated from 0 by γ_1 and γ_2, which we call γ_4. The only nontrivial nonperipheral coponent of $f^{-1}(\gamma_4)$ is separated from 0 by γ and γ_2 and we call it γ_5 Then $f^{-1}(\gamma_5) = \gamma_6$ is connected and has only one nonperipheral perimage γ_7. Components of $f^{-1}(\gamma_7)$ are peripheral. So we have an 8×8 matrix as follows:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The eigenvector with eigenvalue 1 and the smallest possible integer entries is (written as a row vector) $(1,1,1,2,2,1,1,1)$. So $m=0$.

Now we give another example of an s_q with $\mu_q > \mu_{1/3}$ so that $m=0$ for the branch associated to $f = s_{1/3} \amalg s_q$, although the entries of the matrix for f^* are not all integers. We use the convention before on labelling regions L, C, R, and consider the sequence $LRRRCLCLRRCLC$ for $s_q(0)$.



LRRRCLCRRC, $p = 1/3$.

As before let $\gamma_0 = \mu_{1/3} \cup \mu_{1/3}^{-1}$ and let γ_1 be the nonperiodic preimage. Let γ_2 and γ_3 be the preimages of γ_1 with γ_2 separated from 0 by γ_1 and γ_3 separated from 0 by γ_0. Then one component of $f^{-1}(\gamma_2)$ is peripheral and the other is isotopic to γ_3.Then $f^{-1}(\gamma_3)$ is connected and we call this loop γ_4. Both components of $f^{-1}(\gamma_4)$ are peripheral. Indexing the i'th entry by γ_{i-1}, the smallest eigenvector with integer entries (written as a row) is $(1,1,1,2,1)$. So $m=0$.

Strings of examples.

Since we have now considered $LLRCLLRLC$, $p=1/7$ and $((LLRC)^2LLRLC$, $p=1/7$, we can clearly also consider $((LLRC)^mLLRLC$, $p=1/7$. This branch is of multiplicity 2^m and will have knot type (p_m,q_m) where $x_m = p_m/q_m$, $x_{m+1} = \frac{1}{2}(x_m+1)+2$. This gives $x_2 = \frac{1}{2}(\frac{7}{2}+1)+2 = \frac{9}{4}+2 = \frac{17}{4} = \frac{7}{2}+\frac{3}{4}$, suggesting that $x_{m+1} = x_m + 3/2^{m+1}$. Assume true for $m-1$. Then

$$x_{m+1} = \frac{1}{2}(x_m+1)+2 = \frac{1}{2}\left(x_{m-1}+\frac{3}{2^m}\right)+2 = \tfrac{1}{2}(x_{m-1}+1)+2+\frac{3}{2^{m+1}} = x_m + \frac{3}{2^{m+1}}$$

as required. This gives

$$x_m = 2 + \sum_{i=1}^{m} \frac{3}{2^i} = \frac{2^{m+2}+2^m-3}{2^m}.$$

Then the knot type is $(2^{m+2}+2^m-3, 2^m)$.

Another string of examples is given by LLR^tCLLR^tLC, $p = 1/7$. Then $h^{2t+5}_{\zeta,\rho}(1) = h^{t+2}_{\zeta,\rho}(1) + b_1\rho^2 + o(\rho^2)$ for $b_1 \neq 0$. This means that $h^{t+6}_{\zeta,\rho}(1) = h^3_{\zeta,\rho}(1) + b_2\rho^{2t} + o(\rho^{2t})$ for $b_2 \neq 0$. Then $h^{t+4}_{\zeta,\rho}(1) = h_{\zeta,\rho}(1) + b_3\rho^{2t} + o(\rho^{2t})$ for $b_3 \neq 0$. Then $h^{t+3}_{\zeta,\rho}(1) = 1 + b_4\rho^x + o(\rho^x)$ where $b_4 \neq 0$ and $2x-1 = 2t$, that is, $x = t + \frac{1}{2}$. Then $h^2_{\zeta,\rho}(1) = 1 + \sum_{i=1}^{3t} a_i\rho^i + b_5\rho^{3t+1/2} + o(\rho^{3t+1/2})$ for suitable a_i and $b_5 \neq 0$. Then $\zeta = 1 + \sum_{i=1}^{3t} c_i\rho^i + b_6\rho^{3t+1/2} + o(\rho^{3t+1/2})$ for $b_6 \neq 0$. We have $m = 1$. So we know that ζ has a power series expansion in $\rho^{1/2}$. So this gives knot type $(6t+1, 2)$.

We get a similar family by considering $LR^tCLR^tCLR^{t+1}C$. In this case if t is odd, $t = 2r+1$, $r \geq 0$, we get knot type $(14r+9, 2)$.

Arbitrarily many torus iterations.

We can construct knots with arbitrarily many torus iterations by taking X_t, $p = 1/7$, where X_m is the sequence defined by $X_1 = LLRCLLRLC$, $X_{m+1} = X_mX'_m$, where X'_m is obtained from X_m as follows. The sequence X_m always ends either LC or RC. We take X'_m to have all letters the same as X_m, except for the last two, which are either RC or LC, depending on whether the last two letters of X_m are LC or RC. Then the sequence X_m gives a torus knot with m Puiseaux pairs, that is, m successive torus knots.