MATH553. Topology and Geometry of Surfaces
Problem Sheet 6: Lifts, Homotopy, Coverings

Please hand in your solutions in class on *Thursday 17th November.*

1. This question is about determining whether your letter is simply-connected or not. So let S_2 be the subspace of $\mathbb{R}^2$ determined by your letter, which you found in the first CA assignment.

Paul: assuming that the "join" on your letter (K) was $0 \in \mathbb{C}$ show that for all $\lambda < 1$ the map

$$\varphi_\lambda(z) \to \lambda z : \mathbb{C} \to \mathbb{C}$$

maps S_2 into itself and fixes the join point 0. By considering $\varphi_s \circ \alpha$ or otherwise, for any closed loop $\alpha : [0,1] \to S_1$ with $\alpha(0) = \alpha(1) = 0$, show that α s homotopic to a constant path and deduce that S_2 is simply connected. If your join point is a rather than 0, you may instead define φ_λ by

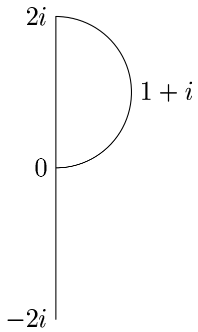
$$\varphi_\lambda(z) = a + \lambda(z - a).$$

Bian Ce, Freddie and Joel: Find a continuous map f from the topological space S_2 determined by your letter to $S^1 = \{z : |z| = 1\}$ in such a way that, for at least one closed path $\alpha : [0,1] \to S_2$, $f \circ \alpha$ is homotopic in S^1 to $t \mapsto e^{2\pi i t}$. Deduce that S_1 is not simply connected. You can probably choose f to be a homeomorphism from some subset of S_2 onto S^1, remembering that a continuous bijection from a Hausdorff space onto a compact space is a homeomorphism.

Example. I am going to do exercises similar to yours with the letter P, realised by

$$S_2 = \{it : t \in [-2,2]\} \cup \{i + e^{it} : t \in [-\pi/2, \pi/2]\}$$

The first part of the exercise will be similar to Paul's, except that this set S_2 is not simply connected, and the second part similar toJoel's, Freddie's and Bian Ce's, the last part similar to Paul's. The letter P is as shown.



Define φ_λ for $1 \geq \lambda \geq 0$by

$$\varphi_\lambda(x+iy) = \begin{matrix} x + iy\lambda & \text{if} & y \leq 0 \\ x + iy & \text{if} & y \geq 0 \end{matrix}$$

The two definitions of φ_λ coincide onthe real axis $\{x+iy : y=0\}$ and φ_λ is continuous and preserves the imaginary axis. In particular the set S_2 is mapped into itself by φ_λ, for any $\lambda \in [0,1]$ and $\varphi_0(S_2)$ is shaped like a letter D. If $\alpha : [0,1] \to S_2$ is any closed path with $\alpha(0) = \alpha(1) = 0$ then $F(t,s) = \varphi_{1-s}(\alpha(t)$ $(s, t \in [0,1])$ is a homotopy between α and a path in $\varphi_0(S_2)$, because $\varphi_1(\alpha(t)) = \alpha(t)$ and $\varphi_s(0) = 0$. If $\varphi_0(S_2)$ were simply connected – which is not the case – we coud deduce that S_2 was simply-connected also.

Now define $f : S_2 \to S^1$ by

$$\begin{array}{lll} f(it) = 1 & \text{if} & t \in [-2,0], \\ f(it) = e^{-i\pi t/2} & \text{if} & t \in [0,2] \\ f(i+e^{it}) = ie^{it} & \text{if} & t \in [-\pi/2, \pi/2], \end{array}$$

Then the three definitions of $f(0)$ all give $f(0) = 1$ and f is continuous because it is continuous on each of the sets $\{it : t \in [0,2]\}$ $\{it : t \in [-2,0]\}$ and $\{2i + 2e^{it} : t \in [-\pi/2, \pi/2]\}$. f certainly maps onto S^1. The circular bit $\{i + e^{it} : t \in [-\pi/2, \pi/2]\}$ of S_2 maps to the upper half of S^1, and $\{ti : t \in [0,2\}$ maps onto top lower half of S^1. f is injective restricted to each of these. So f is a bijection from

$$S_2' = \{it : t \in [0,2]\} \cup \{2i + 2e^{it} : t \in [-\pi/2, \pi/2]\}$$

onto S^1. Since S_2' is Hausdorff and S^1 is compact, f is a homeomorphism between S_2' and S^1. Let α be any closed path in S^1 which is not homtopic to a constant path, e.g. $\alpha(t) = e^{2\pi it}$, $t \in [0,1]$. Then $f^{-1} \circ \alpha$ is a closed path in S_2'. If F is a homotopy between $f^{-1}(\alpha)$ and a constant path in S_2, then

$$G(t,s) = f(F(f^{-1}(\alpha(t)), s))$$

is a homotopy between α and a constant path. Since there is no such homotopy, F does not exist and $f^{-1} \circ \alpha$ is not homotopic to a constant path. So S_2 cannot be simply connected.

2. Find lifts to $\mathbb{R}$ of the following maps $f : S^1 \to S^1$ (using the covering $p : \mathbb{R} \to S^1$ given by $p(\theta) = e^{2\pi i\theta}$):

a) $f(z) = z^4$ b) $f(z) = z^{-1}$, c) $f(z) = iz$,

Now find lifts to $\mathbb{R}^2$ of the following maps of $S^1 \times S^1$, using the covering map $p : \mathbb{R}^2 \to S^1 \times S^1$ given by $p(t,u) = (e^{2\pi it}, e^{2\pi iu})$.

d) $f(z_1, z_2) = (z_2^3, z_1 z_2^2)$, e) $f(z_1, z_2) = (z_1/z_2, z_2^2)$.

3. For each of the maps in question 2, and a lift $\tilde{f}$ find the corresponding group homomorphism $(\tilde{f})_* : \mathbb{Z} \to \mathbb{Z}$ or $(\tilde{f})_* : \mathbb{Z}^2 \to \mathbb{Z}^2$.

[Since $\mathbb{Z}$ and $\mathbb{Z}^2$ are abelian, $(\tilde{f})_*$ is independent of the lift chosen in both cases, and we can write $(\tilde{f})_* = f_*$.]