MATH553. Topology and Geometry of Surfaces
Problem Sheet 2: Quotient Topology

Please hand in your solutions in class on *Thursday 13th October.* Question 3 is part of the assessment on this module. Office hours for this module are now fixed as: Mondays at 3, Tuesdays at 4, Fridays at 11, all in 515, which is reached through 516.

1. Let $X = \mathbb{R}^2 \setminus \{\underline{0}\}$, and define $\sim$ by: $\underline{x} \sim \underline{x}' \Leftrightarrow \underline{x}' = \lambda \underline{x}$ for some $\lambda > 0$. Check that $\sim$ is an equivalence relation on X. Take the usual (subspace) topology on X, and the quotient topology on $X/\sim$. Take the usual (subspace) topology on $S^1 = \{(x,y) : x^2 + y^2 = 1\}$. Find a continuous map $F : X \to S^1$ such that $[F] : X/\sim \to S^1$ is well-defined and a bijection. (It will then automatically be continuous.)

2. Let $X = \mathbb{C} \times \{1,2\}$. Give X the usual topology (as a subspace of $\mathbb{C}^2$) and let $\mathbb{C} \cup \{\infty\}$ be given the 1-point-compactification topology. Let $F : \mathbb{C} \times \{1,2\} \to \mathbb{C} \cup \{\infty\}$ be defined by

$$F(z,1) = z, \ \ F(1/z,2) = z \text{if} z \neq 0, \ \ F(0,2) = \infty.$$

Show that F is continuous.

Hint: all you really need to show is that if $U \subset \mathbb{C}$ is an open set such that $\mathbb{C} \setminus U$ is bounded, then $\{1/z : z \in U\} \cup \{0\}$ is open in $\mathbb{C}$.

Now let the equivalence relation $\sim$ be defined on X by: $(z,j) \sim (z',k) \Leftrightarrow$ either $(z,j) = (z',k)$ or $z' = 1/z$ and $j \neq k$. Check that $\sim$ is an equivalence relation, and show that $[F] : X/\sim \to \mathbb{C} \cup \{\infty\}$ is well-defined, and a bijection. (It is then automatically continuous.)

3. *This problem is part of the CA components of this module and is worth 3 marks.*

For the letter or number you have been given, find a subset S_1 of $\mathbb{R}^2$ which, with the subspace topology, looks like the letter or number. Let

$$S_2 = \cup_{j=1}^{n} [a_j, b_j] \times \{j\} \subset \mathbb{R}^2$$

for some positive integer n, and intervals $[a_j, b_j]$ which you can choose at your convenience. Let S_2 be given the subspace topology, with respect to the standard topology on $\mathbb{R}^2$. Choose an equivalence relation $\sim$ such on S_2 that the quotient space $S_2/\sim$, with the quotient topolology, is homeomorphic to S_1, and find a homeomorphism $G : S_2/\sim S_1$, proving that it is indeed a homeomorphism.

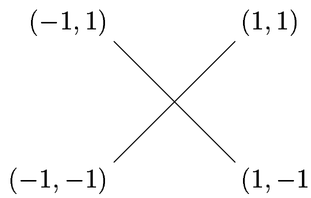
Hint for last part Since S_2 is compact and S_1 is Hausdorff, it suffices to find a map $F : S_2 \to S_1$ which is continuous onto and such that $F(x_1) = F(x_2) \Leftrightarrow x_1 \sim x_2$.

Here is an example of how to tacke this problem for the letter X. There is more that one way to do this, and in fact the sets S_2 and $S_2/\sim$ given below are not the same choices as made in lectures for the symbol + — which is essentially the same as X.

Let

$$S_1 = \{(t,t) \subset \mathbb{R}^2 : -1 \leq t \leq 1\} \cup \{(t,-t) \subset \mathbb{R}^2 : -1 \leq t \leq 1\}.$$

This is a union of two line segments, one from $(-1,-1)$ to $(1,1)$ which lies on the line $x = y$ and the other from $(-1,1)$ to $(1,-1)$ which lies on the line $x+y=0$. The two line segments intersect at $(0,0)$. This certainly looks like the letter X.



Now let

$$S_2 = [-1,1] \times \{1,2\}.$$

Define $\sim$ by $(0,1) \sim (0,2)$, and all other equivalence classes are trivial. Then we claim that $S_2/\sim$, with the quotient topology, is homeomorphic to S_1. Since $S_1 \subset \mathbb{R}^2$, S_1 is Hausdorff, and since S_2 is a closed bounded subset of $\mathbb{R}^2$, S_2 is compact, and the quotient $S_2/\sim$ is compact, because the map $x \mapsto [x] : S_2 \to S_2/\sim$ is continuous onto, where $[x]$ is the equivalence class of x with respect to $\sim$. A continuous map from a compact space to a Hausdorff space which is also a bijection is a homeomorphism. So it suffices to find a continuous bijection $G : S_2/\sim \to S_1$. A continuous surjection $F : S_2 \to S_1$ such that $F^{-1}([x]) = \{y + y \sim x\}$ $(= [x])$ gives rise to a continuous bijection $[F] : S_2/\sim \to S_1$ defined by $[F]([x]) = [F(x)]$, as shown in lectures. So it suffices to find such a continuous surjection F. We define F by

$$F(t,1) = (t,t), F(t,2) = (t,-t) \text{ for all } t \in [-1,1].$$

Then $F : S_2 \to S_1$ is a surjection. Clearly $F(t,1) = F(s,1) \Leftrightarrow s = t$ and $F(t,2) = F(s,2) \Leftrightarrow s = t$. Also $F(t,1) = F(s,2) \Leftrightarrow (t,t) = (s,-s) \Leftrightarrow s = t = 0$. So F has all the required proerties for $[F]$ to be well-defined and a homeomorphism.

4. Let the equivalence relation $\sim$ be defined on $\mathbb{C}$ by $z \sim z' \Leftrightarrow z' = z + m + ni$ for some m, $n \in \mathbb{Z}$. Check that this is an equivalence relation. Fix $\lambda \in \mathbb{C}$ and let $F : \mathbb{C} \to \mathbb{C}$ be given by $F(z) = \lambda z$. Show that $[F] : \mathbb{C}/\sim \to \mathbb{C}/\sim$is well-defined $\Leftrightarrow \lambda = a + ib$ for some a, $b \in \mathbb{Z}$.

Also, determine for which λ $[F]$ is injective.

5. Let $\sim$ be as in question 3. Let $\approx$ be the equivalence realtion defined on $\mathbb{C} \setminus \{0\}$ by $z' \approx z \Leftrightarrow z' = e^{2\pi n} z$ for some $n \in \mathbb{Z}$. Find a continuous map $F : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ such that $z \sim z' \Leftrightarrow F(z) \approx F(z')$, where F is *not* a bijection but $[F] : \mathbb{C}/\sim \to \mathbb{C}/\approx$ *is* .

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Problem Sheet 2: Solutions

1. Take

$$F(\underline{x}) = \frac{\underline{x}}{\|\underline{x}\|}.$$

This is continuous on $\mathbb{R}^2 \setminus \{0\}$, and $F(\underline{x}) = \underline{x}$ if $\underline{x} \in S^1$, so F is surjective. Note that $F(\underline{x}) = F(\underline{y}) \Leftrightarrow \underline{x}/\|\underline{x}\| = \underline{y}/\|\underline{y}\| \Leftrightarrow \underline{x} = \lambda \underline{y}$ with $\lambda = \|\underline{x}\|/\|\underline{y}\|$. Also, $F(\lambda \underline{x}) = \lambda \underline{x}/\|\lambda \underline{x}\| = \underline{x}/\|\underline{x}\| = F(\underline{x})$ for any $\lambda > 0$. So $F(\underline{x}) = F(\underline{y}) \Leftrightarrow \underline{x} \sim \underline{y}$. So $[F]$ is well-defined and injective. $[F]$ is continuous because F is. So $[F]$ is well-defined, continuous and a bijection.

2. A subset U of $\mathbb{C} \times \{1,2\}$ is open $\Leftrightarrow$ the sets $U_j = \{z : (z,j) \in U\}$ are open in $\mathbb{C}$ for $j = 1$, 2. (This is very similar to problem 4 on Sheet 1.) So now let $V \subset \mathbb{C} \cup \{\infty\}$ be open. First suppose that $\infty \notin V$. Then $F^{-1}(V) = (V \times \{1\}) \cup (\{z^{-1} : z \in V\} \times \{2\})$ is open in $\mathbb{C} \times \{1,2\}$ (using that $z \mapsto z^{-1}$ is continuous for $z \neq 0$). Now let $\infty \in V$. Then $\mathbb{C} \setminus V$ is bounded. Then $\{z : |z| > R\} \subset V$ for some $R > 0$. So $\{z : |z| < 1/R\} \subset \{z^{-1} : z \in V\}$. So $\{0\} \cup \{z^{-1} : z \in V\}$ is open. So

$$F^{-1}(V) = ((V \setminus \{\infty\}) \times \{1\}) \cup (\{0\} \cup ((\{z^{-1} : z \in V\}) \times \{2\})$$

is open in $\mathbb{C} \times \{1,2\}$. So F is continuous.

To check $\sim$ is an equivalence relation: it is reflexive because $(z,j) \sim (z,j)$. It is symmetric because if $(z,j) \sim (1/z,k)$ then $(1/z,k) \sim (1/(1/z),j) = (z,j)$ $(j \neq k)$. If $(z,j) \sim (z',k)$ and $(z',k) \sim (z'',\ell)$ then we clearly have $(z,j) \sim (z'',\ell)$ if $(z,j) = (z',k)$ or $(z',k) = (z'',\ell)$. If $(z',k) \neq (z,j)$ and $(z'',\ell) \neq (z',k)$ then $z' = 1/z$, $j \neq k$, $z'' = 1/z'$, $\ell \neq k$, which means that $z'' = z$ and $j = \ell$. So transitivity holds.

From the definition of F, it is clear that $(z,j) \sim (z',k) \Leftrightarrow F(z,j) = F(z',k)$. So $[F]$ is well-defined and injective. Since F is surjective, $[F]$ is too. So $[F]$ is a bijection. Also, since F is continuous, $[F]$ is too.

4. We have $z \sim z + 0 + 0i$, so $\sim$ is reflexive. We have $z' = z + m + ni \Leftrightarrow z = z' - m - ni$, so $\sim$ is symmetric. If $z' = z + m + ni$ and $z'' = z' + p + qi$ $(m,\ n,\ p,\ q \in \mathbb{Z})$ then $z'' = z + (m+p) + (n+q)i$, so $\sim$ is transitive. Suppose that $z' = z + 1$. Then $\lambda.z \sim \lambda.z' = \lambda.z + \lambda \Leftrightarrow \lambda$ is of the form $a + ib$ for some a, $b \in \mathbb{Z}$. Conversely, if $\lambda = a + ib$ for a, $b \in \mathbb{Z}$ then if $z' = z + m + ni$ (any m, $n \in \mathbb{Z}$) then $\lambda.z' = \lambda.z + (am - nb) + i(an + mb)$, so $\lambda.z' \sim \lambda.z$ whenever $z \sim z'$. So $[F]$ is well-defined $\Leftrightarrow \lambda$ is of the form $a + ib$.

For $[F]$ to be injective we need $\lambda.z' \sim \lambda.z \Rightarrow z' \sim z$. Clearly $[F]$ is not injective if $\lambda = 0$. Now $\lambda.(z + (1/\lambda)) = \lambda.z + 1 \sim \lambda.z$ if $\lambda \neq 0$. If $\lambda = a + ib$ for a, $b \in \mathbb{Z}$ then $1/\lambda = a/(a^2+b^2) - ib/(a^2+b^2)$. So for $[F]$ to be injective we need $a/(a^2+b^2) \in \mathbb{Z}$ and $b/(a^2+b^2) \in \mathbb{Z}$. This happens $\Leftrightarrow a^2 + b^2 = 1 \Leftrightarrow \lambda = \pm 1$ or $\pm i$.

5. Define

$$F(z) = e^{2\pi z}.$$

Then $z \sim z' \Leftrightarrow z' = z + m + ni$ for some $m,\, n \in \mathbb{Z} \Leftrightarrow e^{2\pi z'} = e^{2\pi m} e^{2\pi z}$ for some $m \in \mathbb{Z}$, because $e^{2\pi w} = e^{2\pi w'} \Leftrightarrow w' = w + ki$ for some $k \in \mathbb{Z}$. So $[F]$ is well-defined and injective. It is also surjective because F is.