MATH553. Topology and Geometry of Surfaces
Problem Sheet 1: Topology

This module is run both as part of the MMath course, an option which can be taken in eiyther Years 3 and 4, and an M.Sc. course. Lectures are on Mondays at 2, Thursdays at 3 and Fridays at 12. There is a further scheduled contact hour on Monday at 3. I would prefer to have this as an informal office hour, but shall seek the class's opinion on this. The usual day for handing in homework will be Thursdays. It seems sensible to have another office hour in the days before homework hand-in. I suggest either Friday (some time near to the lecture) or Tuesday.

Please hand in your solutions to the following problems in class on *Thursday 6th October* — or earlier if you wish

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1. Show that $[0,1) = \{x : 0 \leq x < 1\}$ is *not* open in $\mathbb{R}$ (in the usual topology). *Hint:* consider the point 0.

2. Show that $U \subset \mathbb{R}^2$ is open in the usual topology on $\mathbb{R}^2$ by finding, for $\overline{x} \in U$, an $\varepsilon > 0$ with $\{\overline{y} : \|\overline{y} - \overline{x}\| < \varepsilon\} \subset U$ where:
a) $U = \{(x,y) \in \mathbb{R}^2 : x > 0\}$, b) $U = \{(x,y) \in \mathbb{R}^2 : x + y > 0\}$.

3. This question is about characterizing open sets in $\mathbb{R}$, in the usual topology. Let U be open in $\mathbb{R}$. For $x \in U$, let

$$I_x = \{y \geq x : [x,y] \subset U\} \cup \{y \leq x : [y,x] \subset U\}.$$

Show that if I_x is not bounded above then $[x,\infty) \subset U$ and that if I_x is not bounded below then $(-\infty, x] \subset U$. Let $b_x = \sup(I_x)$ or ∞, depending on whether or not I_x is bounded above, and similarly let $a_x = \inf(I_x)$ or $-\infty$. Show that $I_x = (a_x, b_x)$ and that a_x, $b_x \notin U$. (Of course, the last statement only makes sense when a_x, $b_x \in \mathbb{R}$.) Finally deduce that U is a *countable* disjoint union of intervals.
Hint: you can consider intervals of the form I_x, and remember that the rationals are countable and dense in $\mathbb{R}$.

4. Take $Y = \mathbb{R} \times \{0,1\} \subset \mathbb{R}^2$, where Y is given the subspace topology. Show that any set of the form $(a,b) \times \{j\}$ (any a, $b \in \mathbb{R}$, and $j = 0$ or 1) is open in Y. Then show that $U \subset Y$ is open in $Y \Leftrightarrow U_j$ is open in $\mathbb{R}$ for $j = 0$ and 1, where

$$U_j = \{x : (x,j) \in U\}.$$

Problem Sheet 1 Solutions

1. There is no $\varepsilon > 0$ with $(-\varepsilon, \varepsilon) \subset [0, 1)$.
2. a) For $(x, y) \in U$, take $\varepsilon = x$. Then if $\|(x', y') - (x, y)\| < \varepsilon$, we have $x|x' - x| < \varepsilon$ and hence $x' > 0$, so that $(x', y') \in U$.
b) For $(x, y) \in U$, take $\varepsilon = (x + y)/2$. Then if $\|(x', y') - (x, y)\| < \varepsilon$, we have $|x - x'| < \varepsilon$ and $|y - y'| < \varepsilon$, so that

$$x' + y' > x + y - |x - x'| - |y - y'| > 0,$$

and $(x', y') \in U$.
3. If I_x is not bounded above, then given $x' \geq x$, there is $y \geq x'$ with $[x, y] \subset U$. So $x' \in U$, for all $x' \geq x$, and so $[x, \infty) \subset U$. Similarly if I_x is not bounded below, $(-\infty, x] \subset U$. Now suppose that I_x is bounded above and $b_x = \mathrm{supI_x}$. Then for all $x' \in [x, b_x)$, there is $y \geq x'$ with $y \in I_x$, that is, $[x, y] \subset U$. So $x' \in U$. So $[x, b_x) \subset U$. If $b_x \in U$ then because U is open, there is $\varepsilon > 0$ such that $(b_x - \varepsilon, b_x + \varepsilon) \subset U$. Then there is $y \in I_x \cap (b_x - \varepsilon, b_x)$, $y \geq x$. Then $[x, b_x + \varepsilon/2] \subset [x, y] \cup (b_x - \varepsilon, b_x + \varepsilon) \subset U$. This means that $b_x + \varepsilon/2 \in I_x$, a contradiction. So $b_x \notin U$.

Similarly, if I_x is bounded below, we have $(a_x, x] \subset U$ and $a_x \notin U$.

Finally, if x, $y \in U$ and $I_x \cap I_y \neq \phi$, then $I_x \cup I_y \subset U$, so $a_x = a_y$ and $b_x = b_y$, that is, $I_x = I_y$ (because $I_x = (a_x, b_x)$). So any two intervals I_x, I_y are either equal or disjoint. The rationals are dense in $\mathbb{R}$. So each interval I_x contains a rational. So there are only countably many such intervals.
4. We have

$$(a, b) \times \{j\} = ((a, b) \times (j - \tfrac{1}{2}, j + \tfrac{1}{2})) \cap (\mathbb{R} \times \{0, 1\}).$$

This is open in Y because $(a, b) \times (j - \frac{1}{2}, j + \frac{1}{2})$ is open in $\mathbb{R}^2$. In fact, more generally, any set $W = V \times (c, d)$ is open in $\mathbb{R}^2$, if V is open in $\mathbb{R}$, because if we take $(x, y) \in W$ then there is $\varepsilon_1 > 0$ with $(x - \varepsilon_1, x + \varepsilon_1) \subset V$ and if

$$\varepsilon = \mathrm{Min}(\varepsilon_1, y - c, d - y)$$

then $\{(x', y') : \|(x', y') - (x, y)\| < \varepsilon\} \subset W$. So W is indeed open.
Now suppose that U is open in $Y = \mathbb{R} \times \{0, 1\}$ and let U_j be as defined. Then if $x \in U_j$, $(x, j) \in U$, so there exists $\varepsilon > 0$ such that

$$\{(x', y') : (x', y') \in \mathbb{R} \times \{0, 1\}\ , \|(x', y') - (x, j)\| < \varepsilon\} \subset U.$$

In particular, if $|x' - x| < \varepsilon$, then $(x', j) \in U$, that is, $x' \in U_j$. So U_j is open in $\mathbb{R}$, for $j = 0$, 1.

Now suppose that $U \subset \mathbb{R} \times \{0, 1\}$ and the sets U_0, U_1 are open in $\mathbb{R}$. Then

$$U = ((U_0 \times (-\tfrac{1}{2}, \tfrac{1}{2})) \cup (U_1 \times (\tfrac{1}{2}, \tfrac{3}{2}))) \cap (\mathbb{R} \times \{0, 1\}),$$

which is open, because as we have already seen the sets $U_0 \times (-\frac{1}{2}, \frac{1}{2})$ and $U_1 \times (\frac{1}{2}, \frac{3}{2})$ are open in $\mathbb{R}^2$.