MATH348. Harmonic Analysis. Problems 8

Work is due in on *Wednesday 24th November.*

1.

Verify that the function

$$u(x,t) = \frac{e^{t-(x^2/4t)}}{\sqrt{t}}$$

satisfies

$$\frac{\partial u}{\partial t} = u + \frac{\partial^2 u}{\partial x^2}.$$

2. Let $u(x,t)$ be continuous and bounded on $\{(x,t) : x \in \mathbf{R}, \mathbf{t} \geq \mathbf{0}\}$. For all $t > 0$ let $u(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$ be defined and integrable in x over $\mathbf{R}$, and let

$$\lim_{x\to\infty} u(x,t) = 0, \quad \lim_{x\to\infty} u_x(x,t) = 0.$$

Let $\hat{u}(\xi,t)$ be the Fourier transform of $u(x,t)$ with respect to x, and let $\hat{u}_x(\xi,t)$ and $\hat{u}_{xx}(\xi,t)$ be similarly defined. Using integration by parts, show that

$$\hat{u}_x(\xi,t) = i\xi\hat{u}(\xi,t), \; \hat{u}_{xx}(\xi,t) = -\xi^2\hat{u}(\xi,t).$$

3. Now let $u(x,t)$ be as in question 2. In addition, for all x, and $t > 0$, let u_t be continuous and *locally uniformly integrable in* x, that is for all $a > 0$, let

$$\sup_{0<t<a} \int_{-\infty}^{\infty} |u_t(x,t)| dx < +\infty$$

and

$$\lim_{\Delta\to\infty} \int_{|x|\geq\Delta} |u_t(x,t)| dx = 0$$

uniformly for $t \in (0,a]$. Let

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u. \tag{2}$$

a) Using question 2, show that

$$\frac{\partial \hat{u}}{\partial t}(\xi,t) = -\xi^2\hat{u}(\xi,t) + \hat{u}(\xi,t).$$

You may assumethat $(\partial\hat{u}/\partial t)(\xi,t)$ is the Fourier transform of $(\partial u/\partial t)(x,t)$ with respect to x. (You will be asked for a step towards this in part b). THe conditions above are neededed for the full result.)) Now solve this differential equation and show that

$$\hat{u}(\xi,t) = \hat{u}(\xi,0)e^{(1-\xi^2)t}.$$

Look in your notes to find a function that $e^{-\xi^2 t}$ is the Fourier transform of (treating t as a constant). Hence show that

$$u(x,t) = \int_{-\infty}^{\infty} u(y,0)\frac{e^{t-(x-y)^2/4t}dy}{2\sqrt{t\pi}}.$$

b) Show that if $h \neq 0$, $t > 0$, $t + h > 0$,

$$\frac{\hat{u}(\xi, t+h) - \hat{u}(\xi, t)}{h} - \hat{u}_t(\xi, t)$$
$$= \frac{1}{h} \int_0^h \int_{-\infty}^{\infty} e^{-i\xi x} (u_t(x, t+y) - u_t(x, t)) dx dy.$$

You should indicate where Tonelli's Theorem is applied. It will help if you can show that

$$\frac{1}{|h|} \int_{[0,h]} \int_{-\infty}^{\infty} |e^{-i\xi x} (u_t(x, t+y) - u_t(x, t))| dx dy \leq 2 \sup_{t'>0} \int_{-\infty}^{\infty} |u_t(x, t') dx.$$

$1a$).
$$\frac{\partial u}{\partial t} = u - \frac{e^{t-(x^2/4t)}}{2t^{3/2}} + \frac{x^2 e^{t-(x^2/4t)}}{4t^{5/2}},$$
$$\frac{\partial u}{\partial x} = \frac{-xe^{t-(x^2/4t)}}{2t^{3/2}},$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{-e^{t-(x^2/4t)}}{2t^{3/2}} + \frac{x^2 e^{t-(x^2/4t)}}{4t^{5/2}}.$$

So
$$\frac{\partial u}{\partial t} = u + \frac{\partial^2 u}{\partial x^2}.$$

2. Using integration by parts,
$$\hat{u}_x(\xi,t) = \lim_{\Delta\to\infty}\int_{-\Delta}^{\Delta} \partial u/\partial x(x,t)e^{-ix\xi}dx = \lim_{\Delta\to\infty}\left[u(x,t)e^{-ixt}\right]_{-\Delta}^{\Delta} + \lim_{\Delta\to\infty}\int_{-\Delta}^{\Delta} u(x,t)i\xi e^{-i\xi x}dx$$
$$= \lim_{\Delta\to\infty}(u(\Delta,t)e^{-i\Delta\xi} - u(-\Delta,t)e^{i\Delta\xi}) + i\xi\hat{u}(\xi,t) = i\xi\hat{u}(\xi,t).$$

Exactly the same calculation with u_x replacing u gives $\hat{u}_{xx}(\xi,t) = i\xi\hat{u}_x(\xi,t)$, and hence
$$\hat{u}_{xx}(\xi,t) = (i\xi)^2\hat{u}(\xi,t) = -\xi^2\hat{u}(\xi,t).$$

3a).Take the Fourier transform with respect to x of the equation (2) of Problems 8. The righthand side becomes $-\xi^2\hat{u}(\xi,t) + \hat{u}(\xi,t)$. So the whole equation becomes
$$\frac{\partial \hat{u}}{\partial t}(\xi,t) = (1-\xi^2)\hat{u}(\xi,t).$$
The solution to an equation
$$\frac{dy}{dt} = (1-\xi^2)y,$$
where ξ is treated as a constant, is
$$y(t) = Ae^{(1-\xi^2)t}$$
for a constant A. Putting in $t=0$ we see that $A = y(0)$ Putting $y(t) = \hat{u}(\xi,t)$, we have
$$\hat{u}(\xi,t) = \hat{u}(\xi,0)e^{t-\xi^2 t}.$$
Now $e^{t-\xi^2 t} = e^t e^{-\xi^2 t}$ and $e^{-\xi^2/2}$ is the Fourier transform of $(1/\sqrt{2\pi})e^{-x^2/2}$, so by question 1 of Problems 6 (for example) $e^{-\xi^2 t}$ is the Fourier transform of $(1/2\sqrt{\pi t})e^{-\xi^2/4t}$. (Alternatively, this particular Fourier transform occurred in working out the solution of the Heat Equation in lectures.) Also, $(f*g)\hat{=}\hat{f}\hat{g}$, and any integrable function is uniquely determined by its Fourier transform. So
$$u(x,t) = \int_{-\infty}^{\infty} u(y,0)\frac{e^{t-(x-y)^2/4t}dy}{2\sqrt{\pi t}}.$$

3b) By definition,

$$\frac{\hat{u}(\xi,t+h)-\hat{u}(\xi,t)}{h}=\int_{-\infty}^{\infty}e^{-i\xi x}\frac{u(x,t+h)-u(x,t)}{h}dx$$

which, by the Fundamental Theorem of Calculus, is equal to

$$\int_{-\infty}^{\infty}\int_0^h e^{-i\xi x}\frac{u_t(x,t+y)}{h}dydx.$$

Also,

$$\hat{u}_t(\xi,t)=\int_{-\infty}^{\infty}\int_0^h e^{-i\xi x}\frac{u_t(x,t)}{h}dydx$$

So

$$\frac{\hat{u}(\xi,t+h)-\hat{u}(\xi,t)}{h}-\hat{u}_t(\xi,t)=\int_{-\infty}^{\infty}\frac{1}{h}\int_0^h e^{-i\xi x}(u_t(x,t+y)-u_t(x,t))dxdy. \quad (1)$$

We want to be able to change the order of integration in (1). We have

$$|e^{-i\xi x}(u_t(x,t+y)-u_t(x,t))|\leq|u_t(x,t+y)|+|u_t(x,t)|.$$

So

$$\begin{aligned}&\frac{1}{|h|}\int_{[0,h]}\int_{-\infty}^{\infty}|e^{-i\xi x}(u_t(x,t+y)-u_t(x,t)))|dxdy\\&\leq\frac{1}{|h|}\int_{[0,h]}\int_{-\infty}^{\infty}|u_t(x,t+y)|+|u_t(x,t)|dxdy\\&\leq 2\frac{|h|}{|h|}\sup_{t'>0}\int_{-\infty}^{\infty}|u_t(x,t')|dx<+\infty.\end{aligned}$$

So Tonelli's theorem applies, and we can change the order of integration in (1) to obtain

$$\frac{\hat{u}(\xi,t+h)-\hat{u}(\xi,t)}{h}-\hat{u}_t(\xi,t)=$$

$$=\frac{1}{h}\int_0^h\int_{-\infty}^{\infty}e^{-i\xi x}(u_t(x,t+y)-u_t(x,t))dxdy.$$