MATH342 Feedback and Solutions 3

1. If n is an even integer then $n \equiv 0 \mod 2$ and $n^k \equiv 0 \mod 2$ that is, n^2 is even. If n is odd then $n \equiv 1 \mod 2$ and $n^k \equiv 1^k \equiv 1 \mod 2$, that is, n^2 is odd.

2. If $n_i \in \mathbb{Z}$ is odd for $1 \leq i \leq k$ then $n_i \equiv 1 \mod 2$ for $1 \leq i \leq k$ and hence $\sum_{i=1}^k n_i \equiv k \mod 2$, so that $\sum_{i=1}^k n_i$ is even if k is odd and odd if k is even.
Alternatively, we can write $n_i = 2m_i + 1$ for some $m_i \in \mathbb{Z}$. Then

$$\sum_{i=1}^k n_i = \sum_{i=1}^k (2m_i + 1) = 2\left(\sum_{i=1}^k m_i\right) + k$$

which is even if k is even and odd if k is odd.

3. We have

$$\frac{p^n - 1}{p - 1} = \sum_{k=0}^{n-1} p^k$$

which is the sum of n odd numbers – which is even if and only if n is even.

Now suppose that $n = 2k$ is even. Then $p^2 - 1 = (p-1)(p+1)$ and

$$\frac{p^{2k} - 1}{p - 1} = (p+1)\sum_{i=0}^{k-1} p^{2i}.$$

The number $p+1$ is even, because p is odd. So $\dfrac{p^{2k}-1}{p-1} \equiv 0 \mod 4$ if and only if either $p+1$ is divisible by 4 or $\displaystyle\sum_{i=0}^{k-1} p^{2i}$ is even. But $p+1$ is divisible by 4 if and only if $p \equiv -1 \mod 4$, and the sum $\sum_{i=0}^{k-1} p^{2i}$ is a sum of k odd numbers, which is even if and only if k is even, that is, if and only if $n+1$ is divisible by 4, that is, if and only if $n \equiv -1 \mod 4$.

Some care is needed to make a complete "if and only if" argument in this case.

If $(p, n) = (3, 1)$ then $3 - 1 = 2$ and $3^2 - 1 = 8$ and $8/2 = 4$ is divisible by 4, which is correct as $n + 1 = 2$ is even and $3 \equiv -1 \mod 4$.
If $(p, n) = (3, 2)$ then $p^3 - 1 = 26$ and $26/2 = 13$ is odd, which is correct as $n + 1 = 3$ is odd.
If $(p, n) = (5, 1)$ then $5 - 1 = 4$ and $5^2 - 1 = 24$ and $24/4 = 6$ is even (divisible by 2) but not divisible by 4, which is correct as $n \equiv 1 \mod 4$ and $p = 5 \equiv 1 \mod 4$.
If $(p, n) = (5, 3)$ then $5^4 - 1 = 624$ and $624/4 = 156 = 4 \times 39$ is divisible by 4, which is correct as $n = 3 \equiv -1 \mod 4$.

This question can be used to deduce that if $N = \displaystyle\prod_{i=1}^k p_i^{n_i}$ *is an odd perfect number and the* p_i *are distinct primes, then* n_i *is odd for exactly one* i*, and, for this* i*,* $p_i \equiv 1 \mod 4$ *and* $n_i \equiv 1 \mod 4$*. This is because exactly one of the numbers* $\dfrac{p_i^{n_i+1} - 1}{p_i - 1}$ *can be even, and none of them can be divisible by 4. These facts are used in question 4.*

4. If $k > 3$ we have

$$\prod_{i=4}^k \left(1 - \frac{1}{p_i}\right) < \prod_{i=4}^k \left(1 - \frac{1}{p_i^{n_i+1}}\right)$$

We also have

$$2\prod_{i=1}^k \left(1 - \frac{1}{p_i}\right) = 2\prod_{i=1}^3 \left(1 - \frac{1}{p_i}\right) \times \prod_{i=4}^k \left(1 - \frac{1}{p_i}\right),$$

and

$$\prod_{i=1}^{k}\left(1-\frac{1}{p_i^{n_i+1}}\right)=\prod_{i=1}^{3}\left(1-\frac{1}{p_i^{n_i+1}}\right)\times\prod_{i=4}^{k}\left(1-\frac{1}{p_i^{n_i+1}}\right).$$

So we have split each product into two parts and we want to compare the two products by comparing the parts. We have compared the products over terms $4 \leq k \leq n$ if $n \geq 4$. Now we need to compare the parts with $1 \leq k \leq 3$.

$$2\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)=\frac{16}{15}\times\frac{6}{7}=\frac{32}{35}=1-\frac{3}{35}<1-\frac{1}{12}.$$

and since $n_1 \geq 2$ $n_2 \geq 1$ and $n_3 \geq 2$,

$$\left(1-\frac{1}{3^{n_1+1}}\right)\left(1-\frac{1}{5^{n_2+1}}\right)\left(1-\frac{1}{7^{n_3+1}}\right)$$

$$\geq\frac{26}{27}\times\frac{24}{25}\times\frac{342}{343}=\frac{208}{225}\times\frac{342}{343}>1-\frac{17}{225}-\frac{1}{343}>1-\frac{1}{13}-\frac{1}{343}$$

Since $\frac{1}{12}-\frac{1}{13}=\frac{1}{156}$, we have

$$2\prod_{i=1}^{3}\left(1-\frac{1}{p_i}\right)<\prod_{i=1}^{3}\left(1-\frac{1}{p_i^{n_i+1}}\right)$$

and hence the two products from 1 to k cannot be equal, and N does not exist.

5. Write

$$\binom{p}{k}=\frac{\prod_{j=0}^{k-1}(p-j)}{k!}.$$

We know that the binomial coefficients are integers and therefore

$$k! \mid \prod_{j=0}^{k-1}(p-j)$$

and

$$\prod_{j=0}^{k-1}(p-j)=q\times k!$$

Since $k>0$, $p \mid \prod_{j=0}^{k-1}(p-j)$ but since $k<p$, p does not divide $k!$. So $p \mid q$, that is

$$p \mid \binom{p}{k}$$

If you use the expression

$$\binom{p}{k}=\frac{p!}{k!(p-k)!}.$$

then you also need to note that $gcd(p,(p-k!)=1$ *for* $1 \leq k \leq p-1$.

To show that is necessary for p to be prime

$$\binom{4}{2}=6$$

is not divisible by 4

6.

a) Since $375=75\times5=15\times5^2=3\times5^3$, there are 75 strictly positive integers ≤ 376 which are divisible by 5, and 15 which are divisible by 5^2 and 3 which are divisible by 5^3. So the maximum power of 5 dividing 376! is $75+15+3=93$. The maximum power of 2 dividing 376 is more than 188, because there are 188 strictly positive even integers which are ≤ 376. So the maximum power of 10 dividing 376! is 93 and there are 93 zeros at the end of the number 376!.

b) We have

$$\binom{376}{128} = \frac{\prod_{k=249}^{376} k}{128!}$$

26 of the integers between 249 and 376 inclusive are divisible by 5, and 6 disible by 5^2 and 2 divisible by 5^3. Meanwhile there are 25 strictly positive integers ≤ 128 which are divisible by 5, and 5 by 5^2, and 1 by 5^3. So the maximum power of 5 dividing $\binom{376}{128}$ is 3. As for the maximum power of 2, there are 64 even integers between 249 and 376 inclusive, and 32 which are divisible by $4 = 2^2$ and 16 which are divisible by $8 = 2^3$. Then 8 of these are divisible by 2^4,, and 4 divisible by 2^5, and 2 divisible by 2^6 and one number, 256 which is divisible by 2^7. But $256 = 2^8$ is also divisible by 2^8. So the power of 2 which divides $\prod_{k=249}^{376} k$ is

$$64 + 32 + 16 + 8 + 4 + 2 + 1 + 1 = 128.$$

As for 128!, there are 2^{7-k} strictly positive integers ≤ 128 which are divisible by 2^k, for each $1 \leq k \leq 7$. So the maximum power of 2 dividing 128 is

$$\sum_{k=0}^{6} 2^k = 127,$$

and the maximum power of 2 dividing $\binom{376}{128}$ is $128 - 127 = 1$. So the maximum power of 10 dividing $\binom{376}{128}$ is 1 and there is just one 0 at the end of this number.

I had some answers to this question which worked entirely with factorials, using $\binom{376}{128} = \frac{376!}{248! \times 128!}$. *This showed initiative, because I had not suggested it myself. But most of the answers that I saw using this failed to separate out powers of* 2 *and* 5 *properly, and so got the calculation wrong. Here is how to do it. We will use* m_1, n_1 *and* k_1 *for the maximumpowers of* 2 *dividing* 376!, 128! *and* 248! *respectively, and* m_2, n_2 *and* k_2 *for the maximum powers of* 5 *dividing* 376!, 128! *and* 248! *respectively. Then*

$$\begin{aligned} m_1 &= \left\lfloor \frac{376}{2} \right\rfloor + \left\lfloor \frac{376}{4} \right\rfloor + \left\lfloor \frac{376}{8} \right\rfloor + \left\lfloor \frac{376}{16} \right\rfloor + \left\lfloor \frac{376}{32} \right\rfloor + \left\lfloor \frac{376}{64} \right\rfloor + \left\lfloor \frac{376}{128} \right\rfloor + \left\lfloor \frac{376}{256} \right\rfloor \\ &= 188 + 94 + 47 + 23 + 11 + 5 + 2 + 1 = 371 \\ n_1 &= \left\lfloor \frac{128}{2} \right\rfloor + \left\lfloor \frac{128}{4} \right\rfloor + \left\lfloor \frac{128}{8} \right\rfloor + \left\lfloor \frac{128}{16} \right\rfloor + \left\lfloor \frac{128}{32} \right\rfloor + \left\lfloor \frac{128}{64} \right\rfloor + \left\lfloor \frac{128}{128} \right\rfloor \\ &= 64 + 32 + 16 + 8 + 4 + 2 + 1 = 127, \\ k_1 &== \left\lfloor \frac{248}{2} \right\rfloor + \left\lfloor \frac{248}{4} \right\rfloor + \left\lfloor \frac{248}{8} \right\rfloor + \left\lfloor \frac{248}{16} \right\rfloor + \left\lfloor \frac{248}{32} \right\rfloor + \left\lfloor \frac{248}{64} \right\rfloor + \left\lfloor \frac{248}{128} \right\rfloor \\ &= 124 + 62 + 31 + 15 + 7 + 3 + 1 = 243 \end{aligned}$$

So the maximum power of 2 *dividing* $\binom{376}{128}$ *is* $m_1 - (n_1 + k_1) = 371 - 127 - 243 = 1$.

For powers of 5*: we have seen already that* $m_2 = 93$. *Similarly we have*

$$n_2 = \left\lfloor \frac{128}{5} \right\rfloor + \left\lfloor \frac{128}{25} \right\rfloor + \left\lfloor \frac{128}{125} \right\rfloor = 25 + 5 + 1 = 31,$$

$$k_2 = \left\lfloor \frac{248}{5} \right\rfloor + \left\lfloor \frac{248}{25} \right\rfloor + \left\lfloor \frac{248}{125} \right\rfloor = 49 + 9 + 1 = 59.$$

So the maximum power of 5 *dividing* $\binom{376}{128}$ *is then* $m_2 - (n_2 + k_2) = 93 - 31 - 59 = 3$. *Since* $1 < 3$, *the maximum power of* 5 *dividing* $\binom{376}{128}$ *is then* 1, *that is, there is just one zero at the end of* $\binom{376}{128}$.