MATH 342 Problem Sheet 8: Groups and Rings

Due Monday 15th April

1. Let H_1 and H_2 be groups. Prove that if $a_1 \in H_1$ has order n_1 and $a_2 \in H_2$ has order n_2, then the order of (a_1, a_2) in $H_1 \times H_2$ is n, where $n = lcm(n_1, n_2)$.

Hence, or otherwise, find all possible orders of elements of G_{56}. You are not requred to find elements of thse orders.

Hint: Remember (from lectures a few weeks ago) that if $n = n_1 \times n_2$, and n_1 and n_2 are coprime, then $G_n \cong G_{n_1} \times G_{n_2}$.

2. Factorise the following as far as possible:

a) $x^4 - 1$ in $\mathbb{Z}_5[x]$;

b) $x^2 + x + 1$ in $\mathbb{Z}_3[x]$.

3. Determine which of the following integers n can be written as a sum of two integer squares. In each case show, from the prime factorisation of n, that it does, or does not, satisfy the criterion for being a sum of two integer squares, and also either produce integers a and b such that $n = a^2 + b^2$, or show that there are no such integers:

$$n = 37, \quad 38, \quad 40,\ 41,\ 44,\ 45.$$

4. Find the prime factorisation in $\mathbb{Z}[x]$ of $x^3 - 1$, $x^4 - 1$, $x^6 - 1$ and $x^{12} - 1$. You will need to check the irreduciblity in $\mathbb{Z}[x]$, of three quadratic polynomials and of one quartic. In the case of the quartic, you will need to check that it has no integer zeros and does not factorise as a product of two quadratics with integer coeffficients.

Hence, or otherwise, determine the cyclotomic polynomials $\psi_d(x)$ for $d = 1$, 2, 3, 4, 6 and 12.

5. Let

$$\mathcal{O}[\sqrt{5}] = \{(c_1 + c_2\sqrt{5}) : (c_1 \in \mathbb{Z} \wedge c_2 \in \mathbb{Z}) \vee (c_1 + \tfrac{1}{2} \in \mathbb{Z} \wedge c_2 + \tfrac{1}{2} \in \mathbb{Z})\}.$$

You may assume that $\mathcal{O}[\sqrt{5}]$ is a ring.

a) Show that $(c_1^2 - 5c_2^2)$ is an integer whenever both $c_1 + \frac{1}{2}$ and $c_2 + \frac{1}{2}$ are both integers.

b) Show that if $n \in \mathbb{Z}$ and $c_1 + c_2[\sqrt{5}]$ divides n in $\mathcal{O}[\sqrt{5}]$ then so does $c_1 - c_2\sqrt{5}$. You may assume that $\theta : \mathcal{O}[\sqrt{5}] \to \mathcal{O}[\sqrt{5}]$ defined by

$$\theta(c_1 + c_2\sqrt{5}) = c_1 - c_2\sqrt{5}$$

is a well-defined ring isomorphism, in particular, that $\theta(cd) = \theta(c)\theta(d)$ for all c and $d \in \mathbb{Z}[\sqrt{5}]$.

c) Show that if $c_1 + c_2\sqrt{5}$ is a unit in $\mathcal{O}[\sqrt{5}]$ – that is, a divisor of 1 – if and only if $c_1 - c_2\sqrt{5}$ is a unit, and if and only if $c_1^2 - 5c_2^2 = \pm 1$.

I will collect solutions at the lecture on Monday 15th April. Any solutions which are not handed in then, or by 5pm that day in the folder outside room 516 will not be marked.