- The properties of the real numbers are fundamental to the development of calculus.
- Yet, to a very large extent, the key properties of the real numbers were not recognised until late in the nineteenth century.
- Examining properties of the real numbers led people to examine the nature of numbers: real, complex, rational and integer.
- Finally, this led to the development of the logical foundations of mathematics, a project which extended into the twentieth century.
- One of the most famous proofs in all mathematics is the proof, found in Euclid, that $\sqrt{2}$ is irrational.
- This provides one of the first examples of a real number which is not rational, that is, not the quotient of one integer by another.
- Legend has it that this proof was found by the Pythagoreans, and that the discovery of non-rational numbers so disrupted the presumed order of things that the discoverer was thrown into the sea.
- How do we think of real numbers? A common non-expert description is as "points on a line".
- We probably think of the real line as having no break in it .
- This, when fully formulated, is, in fact, the property that distinguishes the real numbers from the integers and rational numbers.

The Greek's view of real numbers

- Key properties of the real numbers are identified in Euclid – but in more recent times, the importance was not recognised until the nineteenth cemtury.
- These are nowadays attributed to the Greek mathematician Eudoxus.
- In Euclid, a positive real number is interpreted as a ratio of two lengths.
- If a/b and c/d are two positive real numbers (ratios of lengths a and b, and of c and d respectively) then it is possible to decide which of these is "less than" the other.
- We say that $a/b \leq c/d$ if and only if $ma < nb$ whenever $mc < nd$ for positive integers m and n.
- This is a complete and correct definition of order on positive real numbers.

Theories of the real numbers were presented by:

- William Hamilton,in two papers read to the Irish Academy in 1833 and 1835, but he did not complete the work;

- Weierstrass,in lectures in Berlin in 1859, but he disowned a publication in 1872 which purported to present this theory;
- Méray in 1869;
- Heine in 1870;
- Dedekind, published in 1872, but based on earlier ideas;
- Cantor, published in 1883.
- The best known theories nowadays are those of Dedekind and Cantor.
- Both theories – and indeed any theory – describes the real numbers in terms of the rationals.
- Cantor's description uses *equivalent sequences of rational numbers*, of the type known nowadays as *Cauchy sequences*
- Dedekind's description uses *Dedekind cuts*

A *Dedekind cut* A is a nonempty set of rational numbers with the following properties.

- There is a rational number x such that $x \notin A$.
- If $y \in A$ and $z < y$ is rational, then $z \in A$.
- A has no maximal element.
- The third property was actually left out of Dedekind's description. Some such property is needed. If x is rational we should decide whether $\{y \in \mathbb{Q} : y \leq x\}$ is a Dedekind cut or whether $\{y \in \mathbb{Q} : y < x\}$ is a Dedekind cut, but we should not allow both.
- For example,
  $$\{x \in \mathbb{Q} : x < 0 \text{ or } x^2 < 2\}$$
  is a Dedekind cut —- which we call $\sqrt{2}$.
- A *real number* is then a Dedekind cut.
- Defining arithmetic and order of real numbers is straightforward, *in terms of the arithmetic and order on rational numbers*.
- For example, if A and B are real numbers, then $A + B$ is the Dedekind cut
  $$\{y_1 + y_2 : y_1 \in A, y_2 \in B\}.$$
  It is easy to verify that $A+B$ satisfies the three properties required of a Dedekind cut.
- We have $A < B$ if and only if A is contained in B and $A \neq B$

- Hilbert (1862-1943) gave a list of the axioms of the real numbers, regarding addition, multiplication and order. The list can be found in Kline, pp 990-991.
- But the most important property of real numbers is *completeness*:
- If A_n is a Dedekind cut for every integer $n \geq 1$ and $A_n \subset A_{n+1}$ and there is a rational number x which is not in A_n for any n, then $\cup_{n\geq 1} A_n$ is a Dedekind cut.
- The *Completeness Axiom* is often formulated as:

  if a_n is an increasing (decreasing) sequence of real numbers which is bounded above (below), then $\lim_{n\to\infty} a_n$ exists (as a real number).

How do we know that the rationals exist?

Various people attempted to define and identify and prove properties of the rational numbers:

- Martin Ohm (1792-1872)
- Karl Weierstrass (1815-1897)
- Giuseppe Peano (1858-1932)
- Weierstrass used the description that is used in formal studies today
- The rationals are pairs of integers $[a, b]$ where $b \neq 0$ and where $[a, b] = [c, d]$ if and only if $ad - bc = 0$.
- Also we identify $[a, 1]$ with the integer a.
- We define

  $$[a_1, b_1] + [a_2, b_2] = [a_1 b_2 + a_2 b_1, b_1 b_2],$$

  and

  $$[a_1, b_1] \cdot [a_2, b_2] = [a_1 a_2, b_1 b_2]$$

- The usual rules of arithmetic: associativity, commutativity, distributivity, can be proved from the corresponding rules for the integers, but

What are the integers

- Dedekind published a work called “Was sind die Zahlen”.It was not much read.
- Kronecker said “God made the integers. All else is the work of man”
- The best known axiomatisation of the natural numbers is that of Peano.
- The natural numbers are the positive integers.
- Some people include zero but Peano did not

Peano's axioms

1. 1 is a natural number
2. Every natural number a has a successor $a + 1$
3. 1 is not a successor
4. If $a + 1 = b + 1$ then $a = b$
5. If a set S of natural numbers contains 1 and $a \in S \Rightarrow a + 1 \in S$ then $S = \mathbb{N}$, the set of all natural numbers.

The fifth axiom is what is needed to carry out induction

Addition of integers

- Addition can be defined in terms of successor.
- Addition of a and 1 is just $a + 1$.
- Then if $a + b$ has been defined we define

  $$a + (b + 1) = (a + b) + 1,$$

  that is, the addition of a and the successor of b is defined to be the successor of $a + b$.
- Peano's fifth axiom then gives that addition of a and b is defined for any $a, b \in \mathbb{N}$.

Associativity of addition

- Also we can prove by induction on $c \in \mathbb{N}$ that

  $$a + (b + c) = (a + b) + c$$

  for all $a, b, c \in \mathbb{N}$
- By definition this is true for $c = 1$.
- Suppose it is true for c.
- Then

  $$a + (b + (c + 1)) \; = a + ((b + c) + 1) \; = (a + (b + c)) + 1$$
  $$= ((a + b) + c) + 1 \; = (a + b) + (c + 1)$$

  as required.

But

- How do we know the natural numbers exist?
- We don't of course.We hypothetize
- But can we build up the natural numbers from something simpler?
- Peano's axioms make it clear that the natural numbers are built up from the natural number1
- Or one can use 0 - as is more usually done.
- The approach which developed in the early twentieth century is to identify
  - 0 with the empty set $\emptyset$,
  - 1 with the set containing the empty set $\{\emptyset\}$
  - if the natural number$a$ is a set then $a + 1$ is the set $a \cup \{a\}$.
  - Hence every natural number is a set.
- The properties of the natural numbers therefore depend on the language and axioms of set theory

Set theory

- Cantor had a big role in introducing set theory into mathematics.
- The axiomatization of set theory – and hence of mathematics –was carried out by Bertrand Russell (1872 - 1970) and Alfred North Whitehead (1861-1947)
- Bertrand Russell published his "Principles of Mathematics" in 1903 and together they published ":Principia Mathematica" in 1910-13
- Russell was one of the great figures of the twentieth century: mathematician, philosopher, pacifist in the first world war, educationalist, writer (Nobel prizewinner), founding member of CND.
- Perhaps the most striking example of the need for the Axiomatization of set theory is

Russell's paradox

- Let
$$A = \{x : x \text{ is a set}, x \notin x\}$$
- Is $A \in A$?
- If so then by the definition of A, $A \notin A$, and we have a contradiction.
- If $A \notin A$ then again by the definition of A, we have $A \in A$ which again gives a contradiction.
- So what is wrong?

References

- Kline, M. *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, 1972.
- http://www-history.mcs.st-andrews.ac.uk/history/