Gottfried Wilhelm Leibniz (1646-1716)

- His father, a professor of Philosophy, died when he was small, and he was brought up by his mother.
- He learnt Latin at school in Leipzig, but taught himself much more and also taught himself some Greek, possibly because he wanted to read his father's books.
- He studied law and logic at Leipzig University from the age of fourteen – which was not exceptionally young for that time.
- His Ph D thesis "De Arte Combinatoria" was completed in 1666 at the University of Altdorf. He was offered a chair there but turned it down.
- He then met, and worked for, Baron von Boineburg (at one stage prime minister in the government of Mainz), as a secretary, librarian and lawyer – and was also a personal friend.
- Over the years he earned his living mainly as a lawyer and diplomat, working at different times for the states of Mainz, Hanover and Brandenburg.
- But he is famous as a mathematician and philosopher.
- By his own account, his interest in mathematics developed quite late.
- An early interest was mechanics.
  - He was interested in the works of Huygens and Wren on collisions.
  - He published *Hypothesis Physica Nova* in 1671. The hypothesis was that motion depends on the action of a spirit (a hypothesis shared by Kepler– but not Newton).
  - At this stage he was already communicating with scientists in London and in Paris. (Over his life he had around 600 scientific correspondents, all over the world.)
  - He met Huygens in Paris in 1672, while on a political mission, and started working with him.
  - At Huygens suggestion he started reading the works of St Vincent (a Flemish Jesuit, another key figure in the early development of calculus).
- He also produced a calculating machine in 1670-1, which could carry out the four basic arithmetic operations.
- (In the next decade he developed binary arithmetic.)
- The diplomatic mission to France failed. In 1673 he accompanied von Boineburg's nephew on a related mission to London.

- He came into contact with mathematicians and scientists, including Huygens, while working as an ambassador for the Elector of Mainz, first in Paris, in 1672, and then in London in 1673.
- He visited the Royal Society, was elected a fellow, and talked to a number of scientists there, including Robert Hooke, Boyle and Pell.
- Pell told him that his work on series had been done by a mathematician called Mouton (which was correct).
- Hooke later spoke slightingly of his calculating machine.
- Leibniz returned home and redoubled his efforts in mathematics

Leibniz and calculus

- His notes on calculus date from 1673.
- Many of these were never published. They include original ideas and also his reinterpretation of the works of others.
- Even in his Ph D thesis he was interested in *successive differences* of sequences, and sums of successive differences, that is,

$$a_n = a_0 + (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_n - a_{n-1})$$

- This is the discrete version of the Fundamental Theorem of Calculus.
- In a manuscript in October 1675 he had a statement of the Fundamental Theorem of Calculus:

  "just as $\int$ will increase, so d will diminish the dimensions"
- This was also the manuscript in which he introduced the notation $\int$ for integral – using both this and the *omn* that he had previously used.

Here is an excerpt from this manuscript

- Throughout the 1670's, Leibniz developed his calculus
- By 1676 he had the derivative and integral of x^n.
- In 1677 he had the correct rules for differentiation of sums, products, quotients.
- By 1680 he had the notation dx, dy for differentials.
- His first publications on calculus was in 1684: *Novus Methodus pro maximis et minimis, itemque tangentibus.*.
- Newton heard about Leibniz' work and wrote to him, at least twice, around 1676, to tell him about his own results.

- Both times, Leibniz replied later than Newton expected, simply because the letters took a long time to reach him.
- Newton, however, interpreted this tardiness as meaning that Leibniz wanted to steal his results.
- This was the start of the Newton-Leibniz controversy.
- In a letter to James Bernoulli in 1703, Leibniz describes how his studies in calculus progressed.He mentions many names: Descartes, Cavalieri, Vieta, Huygens, Pascal, Gregory St Vincent, Roberval, James Gregory (but not Newton).
- In 1711 Leibniz was accused of plagiarism in the Transactions of the Royal Society. When he protested, the Royal Society set up a committee to determine priority, but did not ask Leibniz to give evidence. The committee decided in favour of Newton, who wrote the report.
- Leibniz went off to work for the Duke of Hanover (the uncle of George I, later king of Great Britain)
- Among many other activities, he did pioneering work in geology, through planning projects concerning mines in the Harz Mountains.
- He died in obscurity.

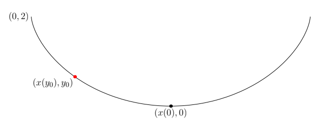
The Bernoulli brothers

- Jakob (James) Bernoulli (1655-1705)
- Johann (John) Bernoulli (1667-1748)
- These brothers were both important mathematicians in their own right and also important correspondents of Leibniz.
- They were among the first readers of Leibniz' work on calculus, and among the first to use the calculus.
- The Bernoulli family produced mathematicians over three generations whose work is still known today.
- They were all called James or John or Daniel or Nicholas. (Since they were Swiss, various versions of their names are used.)
- Although James initially taught John mathematics – in the face of opposition from their father – the brothers were very competitive – and also competitive with Leibniz.
- James had a professorship in Basel.
- John had a chair in Groningen – but in earlier years was paid handsomely by his friend, the mathematician l'Hopital, for teaching him calculus.

- James solved the problem of the *tautochrone* which was also solved by Leibniz.
- John found the solution of the *brachistochrone* problem and issued a challenge to others to find a solution.
- Solutions were found by James Bernoulli, Leibniz, l'Hopital and Newton.

The Tautochrone

- A *tautochrone* or *isochrone* is a monotone curve with a minimum, which can be taken at $y = 0$, such that the time take for a bead to slide along the curve to the bottom is always the same, no matter what the starting point.
- Huygens found that an inverted cycloid is such a curve.



- He tried to make a mechanism to illustrate this but – not surprisingly – it was not possible to eliminate friction, and so he could not do it.
- Some time later James Bernoulli used calculus to verify Huygen's result that the cycloid is the only solution.

How is this done?

- We assume there is no friction.
- So the *potential energy* of a bead of mass m at height y is mgy and the *kinetic energy* is $\frac{m}{2}((dx/dt)^2 + (dy/dt)^2)$ and the sum of these:

$$\frac{m}{2}(2gy + (dx/dt)^2 + (dy/dt)^2)$$

  is constant.
- If the bead starts at height y_0 then the bead is at rest when $y = y_0$, which we can take to happen at $t = 0$,meaning that $x'(0) = y'(0) = 0$ and $y(0) = y_0$. So

$$(x'(t))^2 + (y'(t))^2 = 2g(y_0 - y)$$

- Writing $dx/dt = (dx/dy)(dy/dt)$, we have

$$-\frac{dy}{dt}\sqrt{\frac{(dx/dy)^2 + 1}{2g(y_0 - y)}} = 1.$$

  (Clearly y decreases with t so $dy/dt \leq 0$)

- So
$$\int_0^{y_0} \sqrt{\frac{(dx/dy)^2+1}{2g(y_0-y)}}dy = \int_0^T dt$$
where T is the time taken to slide to the bottom $y=0$.

- The time T is supposed to be the same no matter what the choice of y_0.

- In this integral x is a function of y (not t) so the curve $x(y)$ is the same for all y_0.

- In the integral, write $y = y_0 u$ and write $(dx/dy)(y) = x'(y)$. Then the integral becomes
$$I = \int_0^1 \sqrt{y_0 \frac{((x'(y_0 u))^2+1)}{2g(1-u)}}du,$$
which has to be equal to T for all choices of y_0.

- Since $y=0$ is a minimum of $y(x)$, we expect $dy/dx = 0$ at $y=0$, and therefore we do not expect dx/dy to exist at $y=0$ — and it does not.

- If
$$x'(y))^2 = \frac{A}{y} - 1$$
then
$$1 + (x'(y_0 u))^2 = \frac{A}{y_0}$$
which makes I independent of y_0.

- In fact this is the only way that I can be independent of y_0 (at least if $y(x)$ has a Taylor series expansion).

- So
$$\frac{dx}{dy} = -\sqrt{\frac{A}{y} - 1} = -\sqrt{\frac{A-y}{y}}.$$

So
$$x = -\int \sqrt{\frac{A-y}{y}}dy$$

Making the change of variable $y = A(1+\cos\theta)/2 = A\cos^2(\theta/2)$ gives
$$dy = -(A/2)\sin\theta d\theta, \quad x = A - A\cos^2(\theta/2) = A\sin^2(\theta/2)$$
and
$$x = \int \frac{A}{2}\sqrt{\tan^2(\theta/2)}\sin\theta d\theta = \int A\sin^2(\theta/2)d\theta$$
$$= \int \frac{A}{2}(1-\cos\theta)d\theta = \frac{A}{2}(\theta - \sin\theta)$$
This is indeed the inverted cycloid.

References

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