How to solve a system of linear equations?

What is Linear Algebra?

Linear Algebra develops methods to solve systems of linear equations and tools to analyse such systems of linear equations and their solutions.

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$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

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Linear means that every summand on the left hand side is of the form

$$c \cdot x \quad (c \text{ real constant or coefficient}, \ x \text{ variable}).$$

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$2xy$, $5x^4$, $-\sin x$ are not linear.
If all constants $b_1, \dots, b_m$ on the right hand side are 0 then the system is called homogeneous, otherwise inhomogeneous.

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\begin{array}{ccccccccc}
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a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
\vdots & & & & & & \vdots & & \vdots \\
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\end{array}
$$

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A solution of a system of m linear equations in n variables is a tuple

$$\begin{pmatrix} a_1 & \ldots & a_n \end{pmatrix} \text{ or } \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

of real numbers such that for $x_1 = a_1, \ldots, x_n = a_n$ all m equations hold simultaneously.

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of real numbers such that for $x_1 = a_1, \ldots, x_n = a_n$ all m equations hold simultaneously.
The n-dimensional real vector space $\mathbb{R}^n$ consists of all these "vectors"

$$\begin{pmatrix} a_1 & \ldots & a_n \end{pmatrix}.$$

How to solve a system of linear equations?

1. Extract a matrix
2. Change matrix to (r)REF by row transformations
3. Decide how many solutions exist
4. Calculate solutions

How to solve a system of linear equations?

Step 1: Extract a matrix

What is a matrix?

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A matrix is a rectangular scheme of real numbers, with m rows and n columns.

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How to solve a system of linear equations?

Step 1: Extract a matrix

How to extract a matrix

$$\begin{aligned} x+y+w &= 1 \\ x-w &= 0 \\ x-y+z-w &= 2 \\ x+z+w &= 3 \end{aligned}$$

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Step 1: Extract a matrix

How to extract a matrix

$$
\begin{array}{rcl}
x+y+w & = & 1 \\
x-w & = & 0 \\
x-y+z-w & = & 2 \\
x+z+w & = & 3
\end{array}
\longrightarrow
\begin{array}{ccccccccc}
1\cdot x & + & 1\cdot y & + & 0\cdot z & + & 1\cdot w & = & 1 \\
1\cdot x & + & 0\cdot y & + & 0\cdot z & + & (-1)\cdot w & = & 0 \\
1\cdot x & + & (-1)\cdot y & + & 1\cdot z & + & (-1)\cdot w & = & 2 \\
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$$

1. Insert missing variables multiplied with 0, and mind the signs of the coefficients.

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1. Insert missing variables multiplied with 0, and mind the signs of the coefficients.
2. Strip off coefficients from variables:

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Step 1: Extract a matrix

How to extract a matrix

$$\begin{array}{ccccccccc} 1\cdot x & + & 1\cdot y & + & 0\cdot z & + & 1\cdot w & = & 1 \\ 1\cdot x & + & 0\cdot y & + & 0\cdot z & + & (-1)\cdot w & = & 0 \\ 1\cdot x & + & (-1)\cdot y & + & 1\cdot z & + & (-1)\cdot w & = & 2 \\ 1\cdot x & + & 0\cdot y & + & 1\cdot z & + & 1\cdot w & = & 3 \end{array} \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 & 2 \\ 1 & 0 & 1 & 1 & 3 \end{array}\right)$$

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$$\begin{array}{rcrcrcrcl} 1\cdot x & + & 1\cdot y & + & 0\cdot z & + & 1\cdot w & = & 1 \\ 1\cdot x & + & 0\cdot y & + & 0\cdot z & + & (-1)\cdot w & = & 0 \\ 1\cdot x & + & (-1)\cdot y & + & 1\cdot z & + & (-1)\cdot w & = & 2 \\ 1\cdot x & + & 0\cdot y & + & 1\cdot z & + & 1\cdot w & = & 3 \end{array} \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 & 2 \\ 1 & 0 & 1 & 1 & 3 \end{array}\right)$$

1. Insert missing variables multiplied with 0, and mind the signs of the coefficients.
2. Strip off coefficients from variables:
   - Rows contain coefficients of one equation.

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Step 1: Extract a matrix

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$$\begin{array}{ccccccccc} 1\cdot x & + & 1\cdot y & + & 0\cdot z & + & 1\cdot w & = & 1 \\ 1\cdot x & + & 0\cdot y & + & 0\cdot z & + & (-1)\cdot w & = & 0 \\ 1\cdot x & + & (-1)\cdot y & + & 1\cdot z & + & (-1)\cdot w & = & 2 \\ 1\cdot x & + & 0\cdot y & + & 1\cdot z & + & 1\cdot w & = & 3 \end{array} \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 & 2 \\ 1 & 0 & 1 & 1 & 3 \end{array}\right)$$

1. Insert missing variables multiplied with 0, and mind the signs of the coefficients.
2. Strip off coefficients from variables:
   - Rows contain coefficients of one equation.
   - Columns contain coefficients of one variable.

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Step 1: Extract a matrix

How to extract a matrix

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1. Insert missing variables multiplied with 0, and mind the signs of the coefficients.
2. Strip off coefficients from variables:
   - Rows contain coefficients of one equation.
   - Columns contain coefficients of one variable.
   - Last column contains constants of right hand sides.

How to solve a system of linear equations?

Step 1: Extract a matrix

Notations

$$\begin{aligned} x+y+w &= 1 \\ x-w &= 0 \\ x-y+z-w &= 2 \\ x+z+w &= 3 \end{aligned} \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 & 2 \\ 1 & 0 & 1 & 1 & 3 \end{array}\right)$$

The matrix on the right is called the augmented matrix of the system of linear equations on the left.

How to solve a system of linear equations?

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The matrix on the right is called the augmented matrix of the system of linear equations on the left.
If the system is homogeneous then all all entries in the last column are 0. In that case, we may skip the last column and call the remaining matrix the matrix of a homogeneous system of linear equations.

How to solve a system of linear equations?

Step 2: Change matrix to (r)REF by row transformations

Row transformations

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Row transformations

There are 3 types of row transformations on a matrix

How to solve a system of linear equations?

Step 2: Change matrix to (r)REF by row transformations

Row transformations

There are 3 types of row transformations on a matrix

1. Switching two rows

$$\begin{pmatrix} 2 & 3 & 1 & -2 \\ 3 & 0 & -3 & 6 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 3 & 0 & -3 & 6 \\ 2 & 3 & 1 & -2 \end{pmatrix}$$

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2. Multiplying one row with a non-zero real number

$$\begin{pmatrix} 3 & 0 & -3 & 6 \\ 2 & 3 & 1 & -2 \end{pmatrix} \xrightarrow{\frac{1}{3} \cdot R_1} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 1 & -2 \end{pmatrix}$$

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3. Adding the real multiple of one row to another row

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 1 & -2 \end{pmatrix} \xrightarrow{R_1 - 2 \cdot R_1} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 3 & -6 \end{pmatrix}$$

How to solve a system of linear equations?

Step 2: Change matrix to (r)REF by row transformations

The REF

When is a matrix in REF = Row Echelon Form?

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 3 & -6 \end{pmatrix}$$

Step 2: Change matrix to (r)REF by row transformations

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When is a matrix in REF = Row Echelon Form?

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1. Identify the leading entry = first non-zero entry from the left in each row.

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$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 3 & -6 \end{pmatrix}$$

1. Identify the leading entry = first non-zero entry from the left in each row.
2. Check that the leading entry in every row is to the right of the leading entry in the previous row.

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Graphically we can draw lines in the matrix that separate the leading entries from the zeros in the lower left corner.

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Graphically we can draw lines in the matrix that separate the leading entries from the zeros in the lower left corner.
The matrix is in REF if these lines look like stairs whose steps have all height 1, whereas their depth can be arbitrary.

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Step 2: Change matrix to (r)REF by row transformations

The REF

When is a matrix in REF = Row Echelon Form?

- $\begin{pmatrix} 0 & 3 & 3 & -6 \end{pmatrix}$ in REF?

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Step 2: Change matrix to (r)REF by row transformations

The REF

When is a matrix in REF = Row Echelon Form?

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  Yes – for matrices with only one row we declare step 2 to be satisfied, by convention.

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  Yes – zero rows at the bottom do not destroy the REF property, by convention.

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  Yes – zero rows at the bottom do not destroy the REF property, by convention.
- $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ in REF?

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  Yes, as in the example before.

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  Yes – for matrices with only one row we declare step 2 to be satisfied, by convention.
- $\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ in REF?
  Yes – zero rows at the bottom do not destroy the REF property, by convention.
- $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ in REF?
  Yes, as in the example before. The stairs graphics could be misleading.

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Step 2: Change matrix to (r)REF by row transformations

The rREF

$$\begin{pmatrix} 2 & 3 & 1 & -2 \\ 3 & 0 & -3 & 6 \end{pmatrix}$$

How to solve a system of linear equations?

Step 2: Change matrix to (r)REF by row transformations

The rREF

$$\begin{pmatrix} 2 & 3 & 1 & -2 \\ 3 & 0 & -3 & 6 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 3 & 0 & -3 & 6 \\ 2 & 3 & 1 & -2 \end{pmatrix}$$

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Step 2: Change matrix to (r)REF by row transformations

The rREF

$$\begin{pmatrix} 2 & 3 & 1 & -2 \\ 3 & 0 & -3 & 6 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 3 & 0 & -3 & 6 \\ 2 & 3 & 1 & -2 \end{pmatrix} \xrightarrow{\frac{1}{3} \cdot R_1} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 1 & -2 \end{pmatrix}$$

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Step 2: Change matrix to (r)REF by row transformations

The rREF

$$\begin{pmatrix} 2 & 3 & 1 & -2 \\ 3 & 0 & -3 & 6 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 3 & 0 & -3 & 6 \\ 2 & 3 & 1 & -2 \end{pmatrix} \xrightarrow{\frac{1}{3} \cdot R_1} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 1 & -2 \end{pmatrix}$$

$$\xrightarrow{R_2 - 2 \cdot R_1} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 3 & -6 \end{pmatrix}$$

How to solve a system of linear equations?

Step 2: Change matrix to (r)REF by row transformations

The rREF

$$
\begin{tikzcd}
\begin{pmatrix} 2 & 3 & 1 & -2 \\ 3 & 0 & -3 & 6 \end{pmatrix} \arrow[r, "R_1 \leftrightarrow R_2"] \arrow[d, "\frac{1}{2} \cdot R_1"] & \begin{pmatrix} 3 & 0 & -3 & 6 \\ 2 & 3 & 1 & -2 \end{pmatrix} \arrow[r, "\frac{1}{3} \cdot R_1"] & \begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 1 & -2 \end{pmatrix} \arrow[d, "R_2 - 2 \cdot R_1"'] \\
\begin{pmatrix} 1 & \frac{3}{2} & \frac{1}{2} & -1 \\ 3 & 0 & -3 & 6 \end{pmatrix} & & \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 3 & -6 \end{pmatrix}
\end{tikzcd}
$$

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$$\begin{pmatrix} 2 & 3 & 1 & -2 \\ 3 & 0 & -3 & 6 \end{pmatrix} \xrightarrow{\frac{1}{2}\cdot R_1} \begin{pmatrix} 1 & \frac{3}{2} & \frac{1}{2} & -1 \\ 3 & 0 & -3 & 6 \end{pmatrix} \xrightarrow{R_2 - 3\cdot R_1} \begin{pmatrix} 1 & \frac{3}{2} & \frac{1}{2} & -1 \\ 0 & -\frac{9}{2} & -\frac{9}{2} & 9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 1 & -2 \end{pmatrix} \xrightarrow{R_2 - 2\cdot R_1} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 3 & -6 \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 3 & 1 & -2 \end{pmatrix} \xrightarrow{R_2 - 2 \cdot R_1} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 3 & -6 \end{pmatrix}$$

Fact:

Different row transformations starting on the same matrix may lead to different matrices in REF.

How to solve a system of linear equations?

Step 2: Change matrix to (r)REF by row transformations

The rREF

How to solve a system of linear equations?

Step 2: Change matrix to (r)REF by row transformations

The rREF

$$\begin{pmatrix} 2 & 3 & 1 & -2 \\ 3 & 0 & -3 & 6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 3 & -6 \end{pmatrix}$$

REF, as on p.12

$$\begin{pmatrix} 2 & 3 & 1 & -2 \\ 3 & 0 & -3 & 6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & \frac{3}{2} & \frac{1}{2} & -1 \\ 0 & -\frac{9}{2} & -\frac{9}{2} & 9 \end{pmatrix}$$

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When is a matrix in rREF = reduced REF?

1. The matrix is in REF.
2. Leading entries are 1, and below and above them there are only 0s.

How to solve a system of linear equations?

Step 2: Change matrix to (r)REF by row transformations

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When is a matrix in rREF = reduced REF?

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Fact:

Different row transformations starting on the same matrix always lead to the same matrix in rREF.

How to solve a system of linear equations?

Step 2: Change matrix to (r)REF by row transformations

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Why is this true for any matrix with which we start?

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Different row transformations starting on the same matrix always lead to the same matrix in rREF.

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We need a proof!

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The rREF

Fact:

Different row transformations starting on the same matrix always lead to the same matrix in rREF.

Why is this true for any matrix with which we start?

We need a proof!

But to find such a proof we need to introduce further concepts, so we postpone it to later.

How to solve a system of linear equations?

Step 2: Change matrix to (r)REF by row transformations

Why Step 2?

How to solve a system of linear equations?

Step 2: Change matrix to (r)REF by row transformations

Why Step 2?

Idea: Transforming rows means transforming the corresponding equations!

How to solve a system of linear equations?

Step 2: Change matrix to (r)REF by row transformations

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Idea: Transforming rows means transforming the corresponding equations!

$$\begin{cases} ① & x - z = 2 \\ ② & 2x + 3y + z = -2 \end{cases}$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 2 & 3 & 1 & -2 \end{array}\right)$$

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$$\left\{\begin{array}{lrcr} ① & x-z & = & 2 \\ ② & 2x+3y+z & = & -2 \end{array}\right. \underset{②'+2\cdot①'}{\overset{②-2\cdot①}{\rightleftarrows}} \left\{\begin{array}{lrcr} ①' & x-z & = & 2 \\ ②' & 3y+3z & = & -6 \end{array}\right.$$

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Fact:

Two matrices connected by row transformations correspond to two systems of linear equations with the same solutions.

How to solve a system of linear equations?

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Why is this fact true?

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Suppose that (a, b, c) is a solution of the first system.
Then: $a - c = 2$ and $2a + 3b + c = -2$,

How to solve a system of linear equations?

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Suppose that (a,b,c) is a solution of the first system.
Then: $a-c=2$ and $2a+3b+c=-2$,
hence $a-c=2$ and $(2a+3b+c)-2\times(a-c)=-2-2\times 2$, or $3b+3c=-6$.
So (a,b,c) is a solution of the second system, too.

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hence $a-c=2$ and $(2a+3b+c)-2\times(a-c)=-2-2\times 2$, or $3b+3c=-6$.
So (a,b,c) is a solution of the second system, too.
Similarly, a solution of the second system is a solution of the first system.

How to solve a system of linear equations?

Step 2: Change matrix to (r)REF by row transformations

Why Step 2?

Fact:

Two matrices connected by row transformations correspond to two systems of linear equations with the same solutions.

Consequence

To find the solutions of a system of linear equations we only need to solve the system corresponding to the matrix in REF – and that is much easier, as we will see in Step 4.

How to solve a system of linear equations?

Step 2: Change matrix to (r)REF by row transformations

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Two matrices connected by row transformations correspond to two systems of linear equations with the same solutions.

Consequence

To find the solutions of a system of linear equations we only need to solve the system corresponding to the matrix in REF – and that is much easier, as we will see in Step 4.
If the matrix is in rREF it is even less complicated.

How to solve a system of linear equations?

Step 3: Decide how many solutions exist

No solutions

There are 3 cases, distinguished by the form of the augmented matrix in (r)REF.

How to solve a system of linear equations?

Step 3: Decide how many solutions exist

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Rule 1: No solutions

If the augmented matrix in (r)REF contains a row with the leading entry in the last column then the corresponding system of linear equations has no solutions.

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 4 \end{array}\right)$$

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How to solve a system of linear equations?

Step 4: Calculate the solutions

No solutions

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 4 \end{array}\right)$$

How to solve a system of linear equations?

Step 4: Calculate the solutions

No solutions

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 4 \end{array}\right) \qquad \longleftrightarrow \qquad \begin{aligned} x + 2y &= 1 \\ y + z &= -1 \end{aligned}$$

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The last row is false.
So no matter what numbers we choose for x, y, z this system of linear equations will not be satisfied.

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The last row is false.
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Hence it has no solutions.

Notation

A system of linear equations having no solutions is called inconsistent. If a system of linear equations has solutions it is called consistent.

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How to solve a system of linear equations?

Step 3: Decide how many solutions exist

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Rule 2: Exactly one solution

If every column of the augmented matrix in (r)REF except the last one contains a leading entry then the corresponding system of linear equations has exactly one solution.

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & 4 \end{array}\right)$$

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How to solve a system of linear equations?

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Exactly one solution

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & 4 \end{array}\right) \longleftrightarrow \begin{array}{rcl} x+2y & = & 1 \\ y+z & = & -1 \\ 2z & = & 4 \end{array} \longleftrightarrow \begin{array}{rclcl} x & = & 1-2y & = & 7 \\ y & = & -1-z & = & -3 \\ z & = & 2 & & \end{array}$$

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$$\downarrow \tfrac{1}{2}\cdot R_3$$

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$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\downarrow R_2 - R_3$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

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rREF

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$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

So from a matrix in rREF it is even easier to calculate the solutions.

$$\downarrow R_2 - R_3$$

rREF

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{R_1 - 2\cdot R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right) \longleftrightarrow \begin{array}{rcl} x & = & 7 \\ y & = & -3 \\ z & = & 2 \end{array}$$

How to solve a system of linear equations?

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Infinitely many solution

There are 3 cases, distinguished by the form of the augmented matrix in (r)REF.

How to solve a system of linear equations?

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Infinitely many solution

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Rule 2: Exactly one solution

If there exists a further column of the augmented matrix in (r)REF besides the last one which does not contain a leading entry then the corresponding system of linear equations has infinitely many solution.

$$\left(\begin{array}{ccc|c} 1 & 0 & -4 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

How to solve a system of linear equations?

Step 3: Decide how many solutions exist

Infinitely many solution

There are 3 cases, distinguished by the form of the augmented matrix in (r)REF.

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How to solve a system of linear equations?

Step 4: Calculate the solutions

Infinitely many solutions

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Step 4: Calculate the solutions

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$$\left(\begin{array}{ccc|c} 1 & 0 & 4 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

1. Determine the columns different from the last one that contain no leading entry.

How to solve a system of linear equations?

Step 4: Calculate the solutions

Infinitely many solutions

$$\begin{array}{c} \left(\begin{array}{ccc|c} 1 & 0 & 4 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \\ \begin{array}{cccc} x & y & z & \end{array} \end{array} \longleftrightarrow \begin{array}{rcr} x + 4z & = & 3 \\ y + 2z & = & -1 \\ 0 & = & 0 \end{array}$$

1. Determine the columns different from the last one that contain no leading entry.
2. Recall: Columns of the augmented matrix except the last one belong to variables of the corresponding system of linear equations.

How to solve a system of linear equations?

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1. Determine the columns different from the last one that contain no leading entry.
2. Recall: Columns of the augmented matrix except the last one belong to variables of the corresponding system of linear equations.
   Identify the variables belonging to the columns determined in step 1, and call them free parameters.

How to solve a system of linear equations?

Step 4: Calculate the solutions

Infinitely many solutions

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1. Determine the columns different from the last one that contain no leading entry.
2. Recall: Columns of the augmented matrix except the last one belong to variables of the corresponding system of linear equations. Identify the variables belonging to the columns determined in step 1, and call them free parameters.
3. Express variables that are not free parameters in terms of free parameters.

How to solve a system of linear equations?

Step 4: Calculate the solutions

Infinitely many solutions

$$\begin{array}{c} \left(\begin{array}{ccc|c} 1 & 0 & 4 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \\ \begin{array}{ccc} x & y & z \end{array} \end{array} \longleftrightarrow \begin{array}{rcl} x+4z & = & 3 \\ y+2z & = & -1 \\ 0 & = & 0 \\ \multicolumn{3}{c}{z \text{ is free parameter}} \end{array} \Longrightarrow \begin{array}{rcl} x & = & 3-4z \\ y & = & -1-2z \end{array}$$

What are the solutions, really?

How to solve a system of linear equations?

Step 4: Calculate the solutions

Infinitely many solutions

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z is free parameter

What are the solutions, really?

Vectors $(3-4z, -1-2z, z)$ in $\mathbb{R}^3$ for any $z \in \mathbb{R}$!

How to solve a system of linear equations?

Step 4: Calculate the solutions

Infinitely many solutions

$$
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\end{array}
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What are the solutions, really?

Vectors $(3-4z, -1-2z, z)$ in $\mathbb{R}^3$ for any $z \in \mathbb{R}$!

That means: We can replace z by any real number and obtain a solution – that is why z is called free parameter.

How to solve a system of linear equations?

Step 4: Calculate the solutions

Infinitely many solutions

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What are the solutions, really?
Vectors $(3-4z,-1-2z,z)$ in $\mathbb{R}^3$ for any $z \in \mathbb{R}$!
That means: We can replace z by any real number and obtain a solution – that is why z is called free parameter.
For example, if $z=1$ we obtain the solution $(-1,-3,1)$.

How to solve a system of linear equations?

Step 4: Calculate the solutions

Infinitely many solutions

$$\begin{array}{c} \left(\begin{array}{ccc|c} 1 & 0 & 4 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \\ \begin{array}{ccc} x & y & z \end{array} \end{array} \longleftrightarrow \begin{array}{rcl} x+4z &=& 3 \\ y+2z &=& -1 \\ 0 &=& 0 \end{array} \Longrightarrow \begin{array}{rcl} x &=& 3-4z \\ y &=& -1-2z \end{array}$$

z is free parameter

What are the solutions, really?

Vectors $(3-4z, -1-2z, z)$ in $\mathbb{R}^3$ for any $z \in \mathbb{R}$!

That means: We can replace z by any real number and obtain a solution – that is why z is called free parameter.

For example, if $z = 1$ we obtain the solution $(-1, -3, 1)$.

Indeed, we can check this on the original system of linear equations:

$$\begin{array}{rcl} -1+4\times 1 &=& 3 \\ -3+2\times 1 &=& -1 \end{array}$$

How to solve a system of linear equations?

1. Extract a matrix
   - What is a matrix?
   - How to extract the matrix
   - Notations
2. Change matrix to (r)REF by row transformations
   - Row transformations
   - The REF
   - The rREF
   - Why Step 2?
3. Decide how many solutions exist
4. Calculate solutions
   - No solutions: inconsistent system of linear equations
   - Exactly one solution
   - Infinitely many solutions