Conformal Mappings of Elementary Holomorphic Functions

MATH206 Project A for MATH243)
2009/10 academic year

Sources (the overlap of the two sources is non-empty)**:**

[P] H. A. Priestley, Introduction to Complex Analysis, Clarendon Press, Oxford, 1995, pages 13–16, 88–89, 168–180
(Copies can be requested).

[M] A. I. Markushevich, Theory of Functions of a Complex Variable, vol.I, Prentice-Hall, 1965, pages 118–124, 132–135, 140–146, 150–157, 168–183, 197–207; optional pages 212–237 (Copies provided).

In all the problems below, the theoretical piece and tasks under the same letter are to be done by one student.

In all the problems, $z = x + iy$ and $w = u + iv$ denote respectively the source and target variables, with x, y, u, v all real. Solutions to some problems are not unique, but all the tasks in these problems have more or less unique best (that is, simplest) solutions.

THEORY

A. Complex differentiation and the Cauchy-Riemann equations.
[P], Sections 2.1–2.5.
Geometric interpretation of the derivative. Conformal mappings.
[P], Sections 10.6–10.7;
[M], Sections 31, 32, 33.

B. Elementary entire function $(z - a)^n$.
[P], Sections 10.15–10.16;
[M], Sections 37, 39 (pp 144–146 optional); Optional Sections 41, 42.

C. Elementary entire function e^z.
[P], Sections 10.15–10.16;
[M], Sections 37, 39 (pp 144–146 optional); Optional Sections 41, 42.

D & H. Elementary meromorphic functions: Möbius transformations.
[P], Sections 10.9–10.14, 10.17;
[M], Sections 33, 45, 46, 47, 49, 51; Optional Sections 48, 52.
On the extended complex plane see [P], Section 6.13.

E & F. Elementary meromorphic functions: the Joukowski function.
[P], Sections 10.9–10.14, 10.17;
[M], Sections 33, 45, 46, 47, 49, 51; Optional Sections 48, 52.
On the extended complex plane see [P], Section 6.13.

G. Constructing conformal mappings. The Riemann mapping theorem.
[P], Sections 10.8, 10.14, 10.18–10.21.

PROBLEMS — A

Problem 1. Consider the function

$$w = \frac{1}{2}tz^2 - \frac{1}{4}iz^4,$$

where $t \geq 0$ is a fixed non-negative real number. Find the rotation angle of a direction emerging from point z_0 and the magnification ratio at z_0 if $z_0 = -1$.

Problem 2. Determine on which part of the plane the function of complex variable z

$$w = \frac{tz}{z+1}$$

is stretching and on which it is shrinking. Here $t > 0$ is a fixed positive real number.

Problem 3. Describe the inverse image of the region

$$\{w = u + iv : 0 < \arg w < 3\pi/2,\ |w| < 27\}$$

under the function $w = (z+i)^3$.

Problem 4. Construct a one-to-one mapping which sends the strip S to the sector C:

$$S = \{y < x < y + 1\}, \quad C = \{0 < \arg w < \pi/3\}.$$

Hint: first transform z linearly and apply an exponential map afterwards.

Problem 5. Find the image of the domain

$$\{z : -3 < y < -2\}$$

under the Möbius transformation

$$w = \frac{iz}{z+2}.$$

Problem 6. Map one-to-one onto the upper half-plane $v > 0$ the region

$$\{z : |z+i| < 1, |z-i| > 2\}.$$

PROBLEMS — B

Problem 1. Determine on which part of the plane the function of complex variable z

$$w = tz^2 - 2iz$$

is stretching and on which it is shrinking. Here t is any fixed real number.

Problem 2. Describe the inverse image of the region

$$\{w : u \geq 0, v < 0, |w| < 1/8\}$$

under the function $w = z^3$.

Problem 3. Construct a one-to-one mapping which sends the strip S to the sector C:

$$S = \{-2x - \sqrt{5} < y < -2x\}, \quad C = \{-\frac{\pi}{3} < \arg w < \frac{\pi}{3}\}.$$

Hint: first transform z linearly and apply an exponential map afterwards.

Problem 4. Find the image of the domain

$$\{z : 0 < y < t\}$$

under the Möbius transformation

$$w = \frac{z+i}{z-2i}.$$

Here $t > 0$ is any fixed positive real number.

Problem 5. Let

$$J(z) = (z + z^{-1})/2$$

be the Joukowski function. Basic hyperbolic and trigonometric functions are compositions of the exponential and Joukowski functions, and, if necessary, linear transformations. Check the formula

$$\cosh z = J(e^z).$$

Using this representation and applying appropriate sequence of transformations, obtain a description of the image of the region

$$\{z : -\frac{\pi}{3} < x < \frac{\pi}{3}\}$$

under this map.

Problem 6. Map one-to-one onto the upper half-plane $v > 0$ the region

$$\{z : |z| < 1, |z - 3 - 4i| < 5\}.$$

PROBLEMS — C

Problem 1. Determine on which part of the plane the function of complex variable z

$$w = \frac{iz}{z+t}$$

is stretching and on which it is shrinking. Here t is any fixed real number.

Problem 2. Describe the inverse image of the region

$$\{w : -\frac{\pi}{2} < \arg w < \frac{\pi}{3}, |w| < 8\}$$

under the function $w = z^3$.

Problem 3. Construct a one-to-one mapping which sends the strip S to the sector C:

$$S = \{x < \sqrt{3}y < x + t\}, \quad C = \{-\frac{\pi}{3} < \arg w < \frac{\pi}{2}\},$$

where $t > 0$ is any fixed positive real number. Hint: first transform z linearly and apply an exponential map afterwards.

Problem 4. Find the image of the domain

$$\{z : x > 0, y < t\}$$

under the Möbius transformation

$$w = \frac{z-i}{z+i}.$$

Here t is any fixed real number.

Problem 5. Find the Möbius transformation which carries the points

$$1, i, -i$$

to the points

$$-i, 1, \infty.$$

To which point this Möbius transformation maps 0?

Problem 6. Map one-to-one onto the upper half-plane $v > 0$ the region

$$\{z : |z+1| < 1, |z+i| < 1\}.$$

PROBLEMS — D

Problem 1. Consider the function of complex variable z

$$w = e^{tz},$$

where $t > 0$ is any fixed positive real number. Find the rotation angle of a direction emerging from point z_0 and the magnification ratio at z_0 if $z_0 = -1 + i$.

Problem 2. Determine on which part of the plane the function of complex variable z

$$w = \frac{z - it}{z + it}$$

is stretching and on which it is shrinking. Here $t > 0$ is any fixed positive real number.

Problem 3. Describe the inverse image of the region

$$\{w : -\frac{2\pi}{3} < \arg w < 0, |w| > 4\}$$

under the function $w = z^4$.

Problem 4. Construct a one-to-one mapping which sends the strip S to the sector C:

$$S = \{1 < x < t\}, \quad C = \{0 < \arg w < \frac{\pi}{2}\}.$$

Here $t > 1$ is any fixed real number strictly higher than 1.

Hint: first transform z linearly and apply an exponential map afterwards.

Problem 5. Find the Möbius transformation which carries the points

$$i, a, 1 + i$$

to the points

$$\infty, i, 0,$$

where $a \neq i, 1 + i$ is any fixed point of the extended complex plane except for i and $1 + i$.

Problem 6. Map one-to-one onto the lower half-plane $v > 0$ the region

$$|z - 1 - i| < 1, |z + 1 - i| > 2.$$

PROBLEMS — E

Problem 1. Determine on which part of the plane the function of complex variable z

$$w = tz^2 + z$$

is stretching and on which it is shrinking. Here t is any fixed real number.

Problem 2. Describe the inverse image of the region

$$\{w : \frac{2\pi}{3} < \arg w < \frac{4\pi}{3}, |w| > 16\}$$

under the function $w = z^2$.

Problem 3. Construct a one-to-one mapping which sends the strip S to the sector C:

$$S = \{-\sqrt{5}x - t < y < -\sqrt{5}x\}, \quad C = \{-\frac{\pi}{6} < \arg w < \frac{\pi}{6}\},$$

where $t > 0$ is any fixed positive real number.

Hint: first transform z linearly and apply an exponential map afterwards.

Problem 4. Find the Möbius transformation which carries the points

$$1 - i, a, 1$$

into the points

$$\infty, 1, a,$$

where $a \neq 1, 1 - i$ is any fixed complex number, different from 1 and $1 - i$.

Problem 5. Let

$$J(z) = (z + z^{-1})/2$$

be the Joukowski function. Basic hyperbolic and trigonometric functions are compositions of the exponential and Joukowski functions, and, if necessary, linear transformations. Check the formula

$$\sinh z = iJ(e^{z - i\pi/2}).$$

Using this representation and applying the appropriate sequence of transformations, obtain a description of the image of the region

$$\{z : x < 0, -\frac{\pi}{4} < y < \frac{\pi}{4}\}$$

under this map.

Problem 6. Map one-to-one onto the upper half-plane $v > 0$ the region

$$\{z : |z + i| < 2, |z| < 2\}.$$

PROBLEMS — F

Problem 1. Consider the function

$$w = iz - z^2.$$

Find the rotation angle of a direction emerging from point z_0 and the magnification ratio at z_0 if $z_0 = i + t$, where t is any fixed real number.

Problem 2. Describe the inverse image of the region

$$\{w : -\frac{\pi}{2} < \arg w < \frac{\pi}{2}, |w| < 1/4\}$$

under the function $w = z^6$.

Problem 3. Construct a one-to-one mapping which sends the strip S to the sector C:

$$S = \{-\sqrt{2}x + \frac{\pi}{2} < y < -\sqrt{2}x + \pi\}, \quad C = \{0 < \arg w < \frac{\pi}{2}\}.$$

Hint: first transform z linearly and apply an exponential map afterwards.

Problem 4. Find the image of the domain

$$\{0 < \arg z < \frac{\pi}{3}\}$$

under the Möbius transformation

$$w = \frac{tz + i}{z},$$

where $t > 0$ is any fixed positive real number.

Problem 5. Let

$$J(z) = (z + z^{-1})/2$$

be the Joukowski function. Basic hyperbolic and trigonometric functions are compositions of the exponential and Joukowski functions, and, if necessary, linear transformations. Check the formula

$$\cos z = J(e^{iz}).$$

Using this representation and applying the appropriate sequence of transformations, obtain a description of the image of the region

$$\{z : |x| < \frac{\pi}{6}, y < 0\}$$

under this map.

Problem 6. Map one-to-one onto the upper half-plane $v > 0$ the region

$$\{z : |z| > |z - 2|, |z - 1| < 1\}.$$

PROBLEMS — G

Problem 1. Determine on which part of the plane the function of complex variable z

$$w = -tz^2 + 2iz$$

is stretching and on which it is shrinking. Here t is any fixed real number.

Problem 2. Describe the inverse image of the region

$$\{w = u + iv : u \geq 0,\ v > 0,\ |w| \leq 16\}$$

under the function $w = iz^4$.

Problem 3. Construct a one-to-one mapping which sends the strip S to the sector C:

$$S = \{x < ty < x + 1\}, \quad C = \{-\frac{\pi}{3} < \arg w < \frac{\pi}{3}\}.$$

Here $t > 0$ is any fixed positive real number.

Hint: first transform z linearly and apply an exponential map afterwards.

Problem 4. Find the image of the domain

$$\{z : x > -1, y > 0\}$$

under the Möbius transformation

$$w = \frac{z - it}{z - t}.$$

Here $t > 0$ is any fixed positive real number.

Problem 5. Let

$$J(z) = (z + z^{-1})/2$$

be the Joukowski function. Basic hyperbolic and trigonometric functions are compositions of the exponential and Joukowski functions, and, if necessary, linear transformations. Check the formula

$$\sin z = J(e^{i(z-\pi/2)}).$$

Using this representation and applying appropriate sequence of transformations, obtain a description of the image of the region

$$\{z : |x| < \frac{\pi}{2}, y > 0\}$$

under this map.

Problem 6. Map one-to-one onto the upper half-plane $v > 0$ the region

$$\{z : |z| < 3, |z - 2| < 2\}.$$

PROBLEMS — H

Problem 1. Determine on which part of the plane the function of complex variable z

$$w = \frac{z+t}{z-1}$$

is stretching and on which it is shrinking. Here t is any fixed real number.

Problem 2. Describe the inverse image of the region

$$\{w : -\frac{3\pi}{4} < \arg w < \frac{3\pi}{4}, |w| \geq 5\}$$

under the function $w = z^3$.

Problem 3. Construct a one-to-one mapping which sends the strip S to the sector C:

$$S = \{x < 2y < x + t\}, \quad C = \{-\frac{\pi}{4} < \arg w < \frac{\pi}{3}\},$$

where $t > 0$ is any fixed positive real number. Hint: first transform z linearly and apply an exponential map afterwards.

Problem 4. Find the image of the domain

$$\{z : x > 1,\ y > 1\}$$

under the Möbius transformation

$$w = \frac{z+t}{z-i}.$$

Here $t > 0$ is any fixed positive real number.

Problem 5. Find the Möbius transformation which carries the points

$$i, \infty, 1-i$$

to the points

$$\infty, 1, 1+i.$$

To which point this Möbius transformation maps $-i$?

Problem 6. Map one-to-one onto the upper half-plane $v > 0$ the region

$$\{z : |z| < 2, |z - i - \sqrt{3}| < 2\}.$$

ORGANIZATIONAL MATTERS

The members of the group should distribute the letters between themselves. The student to whom, say, the letter A has been assigned, is responsible for presenting the theoretical piece A at both presentations and also for writing up this piece. All the problems under the same letter A are to be done by this student. The same applies to B,C,. . . .

PRESENTATIONS

The 1st presentation (week 5 approximately) is assumed to be based on a considerable part of the theory involved.

The final presentation should give ideas of solving tasks from each of the problems.

WRITING UP

Writing up the introduction could either be shared or done by one person. The structure of the written presentation should be as follows: the student responsible for the letter A writes up the corresponding theory (definitions, theorems with or without proofs) and solutions of the A-problems. The same applies to the other letters.

The final text should look as follows: first theoretical pieces (A,. . .), after that the solutions to the problems, first A, then B, C etc.