Iteration and Fixed Points

MATH206 Project (after MATH241)

The included notes were taken from a variety of sources, but before reading them, you are advised to refresh your memory of important ideas from MATH241:

- points and their stability;
- iteration and sequences;

Also, it would be a good idea to recall concepts from other modules:

- proof by mathematical induction;
- complex numbers (from MATH103);
- inverse functions;
- features of graphs of functions (maxima etc.)

Sources

[B-G-R] J.W. Bruce, P.J. Giblin and P.J. Rippon, Micro Computers and Mathematics, pp 345-362, pp55-62.

[D] R. Devaney, An Introduction to Chaotic Dynamical Systems, pp 24-31, pp 60-70

[E] Elaydi, Discrete Chaos, pp 289-301, pp 20-29, pp 51-70, pp 135-146

[O] O' Neil, Advanced Engineering Mathematics, pp 729-735

In all the problems below, the theoretical piece and tasks under the same letter are to be done by one student.

Theory

A. Iterative sequences and iteration under Möbius transformations.
$[B - G - R]$, pp345-351.

B. Iteration under Möbius transformations and quadratic polynomials.
[E], pp289-301.

C. The logistic map.
$[B - G - R]$, pp345-347, 352-362;

D. Stability of fixed points.
[E], 20-29;
[D], 69-70.

E. Periodic points and Singer's Theorem.

[D], 24-31 and 69-70; [E], 62-68;

F . Iteration of matrices.
[E], 135-146;

G. Newton's method

[E], 21-23, [B – G – R], 55-62.

H. One-dimensional dynamics

[D], 60-68, [E], 51-60.

Exercises for Section A

a. Find an explicit solution for the iterative sequence

$$x_{n+1} = x_n^2, \quad n = 0,\ 1,\ 2, \cdots$$

with initial term x_0. Consider the sequence

$$x_{n+1} = x_n^2 + 2x_n, n = 0, 1, 2, \cdots$$

with initial term x_0. By using the change of variables $x = u - 1$, show that

$$x_n = (x_0 + 1)^{2^n} - 1, \quad n = 0,\ 1,\ 2, \cdots$$

b (i). Suppose that f, g are conjugate functions with

$$g = \phi^{-1} \circ f \circ \phi.$$

Show that if c is a fixed point of f, then $\phi^{-1}(c)$ is a fixed point of g.

(ii). Do exercise 8.1 from section A, p.349

The following questions are concerned with Möbius sequences $x_{n+1} = f(x_n)$, *where*

$$f(x) = \frac{ax + b}{cx + d}, \quad c \neq 0, \quad ad - bc \neq 0$$

for real constants a, b, c, d.

c (i). Determine the fixed points of f. Show that if

$$(a - d)^2 + 4bc > 0$$

then f has two distinct fixed points, called α and β, say.

(ii). Show that

$$(c\alpha + d)(c\beta + d) = ad - bc,$$

and, using this, show that $c\alpha + d \neq 0$ and $c\beta + d \neq 0$.

***d (i)** Use the facts that $\alpha = f(\alpha)$ and $\beta = f(\beta)$ to verify the equation

$$\frac{x_{n+1} - \alpha}{x_{n+1} - \beta} = \left(\frac{c\beta + d}{c\alpha + d}\right)\left(\frac{x_n - \alpha}{x_n - \beta}\right), \; x_n \neq \beta. \qquad (1)$$

(ii). Deduce from d (i) that

$$\frac{x_n - \alpha}{x_n - \beta} = \left(\frac{c\beta + d}{c\alpha + d}\right)^n \left(\frac{x_0 - \alpha}{x_0 - \beta}\right), \; n = 0, \; 1, \; 2, \cdots$$

e (i). Show that

$$f'(\alpha) = \frac{c\beta + d}{c\alpha + d} \quad \text{and} \quad f'(\beta) = \frac{c\alpha + d}{c\beta + d},$$

and deduce that, if $|f'(\alpha)| < 1$, then $x_n \to \alpha$ as $n \to \infty$ for all real numbers x_0, apart from β and $-d/c$.

(ii). Using the iteration $x_{n+1} = f(x_n)$ in the case of $a = 1$, $b = 2$, $c = d = 1$, take $x_0 = 0$ and iterate until you obtain x_6 up to 4 decimal places. Verify that $x_6 = \sqrt{2}$ up to 4 decimal places. Why does this illustrate the result of **e (i)**?

***f.** If $(a - d)^2 + 4bc = 0$, then there is only one fixed point of f. In this case, show that

$$(a + d)^2 = 4(ad - bc),$$

and deduce that $f'(\alpha) = 1$. Show, further, that the change of variables

$$u = \frac{1}{x - \alpha}$$

transforms equation (1) of **d (i)**into

$$u_{n+1} = u_n + \frac{2c}{a + d}.$$

Deduce that $u_n \to \pm\infty$, and hence that $x_n \to \alpha$ as $n \to \infty$, in this case.

Exercises for Section B

***a.** Prove that any Möbius transformation can be written as a composition of the three following forms of maps:

$$z \mapsto z + \lambda, \; \lambda \in \mathbb{C},$$
$$z \mapsto \frac{1}{z},$$
$$z \mapsto \mu z, \quad \mu \in \mathbb{C}.$$

b. Obtain the fixed points of T, and, if possible, use Theorem 7.1 from section B, p. 296, to determine their stability in these cases:

(i). $a = 1 - 2i$, $b = c = 0$, $d = 1$;

(ii). $a = i$, $b = 1/4$, $c = 2i$, $d = 1$.

For questions **c**, **d**, *let T be such that $a = 1$, $b = i$, $c = 1$, $d = -i$.*

***c.** Show that the fixed points of T in this case are

$$\frac{(1+i)(1 \pm \sqrt{3})}{2}.$$

d. Obtain the derivative of T at its fixed points. Deduce that Theorem 7.1, p.296, cannot be used to determine their stabilty.

***e.** Consider the map $Q_{1/2}(z) = z^2 + 1/2$.

> **(i.** Obtain the fixed points of $Q_{1/2}$, and determine their stability.
>
> **(ii).** Find the 2-cycles of $Q_{1/2}$.

***f.** Show that if $|1 + c| < 1/4$, then $Q_c(z) = z^2 + c$ has an attracting 2-cycle.

Exercises for Section C

These exercises are concerned with the logistic sequence given by $x_{n+1} = f_\lambda(x_n)$ where $f_\lambda(x) = \lambda x(1-x)$ for $0 < \lambda \leq 4$.

***a.** Familiarize yourself with Dr Toby Hall's *Iterator* program on

http://www.liv.ac.uk/∼tobyhall/math206/

which contains full instructions. The program computes iterations of the functions f (but uses a as the parameter rather than λ) and then draws a spider diagram. Consider the following intervals of values of λ:

(i) $0 \leq \lambda \leq 1$;
(ii) $1 < \lambda \leq 2$;
(iii) $2 < \lambda \leq 3$;
(iv) $3 < \lambda \leq 3.45$ (approx);
(v) $\lambda > 3.45$ (approx).

For each of the intervals **(i)** to **(iv)**, choose a value of λ and use *Iterator* to produce and save a picture starting with $x_0 = 0.3$. For interval **(v)**, produce pictures for $\lambda = 3.5$, 3.56, 3.58, 3.7, 4. (For intervals **(i)**, **(ii)**, **(iii)**, have *initial iterations* = 0; for intervals **(iv)**, **(v)** have *initial iterations* = 100.) Include these pictures in your written work by following the on-screen instructions. then describe what happens to the sequence $x_{n+1} = f(x_n)$ as n gets large for each of the intervals **(i)** to **(v)**.

b. Do exercise 9.1 of section C, p. 352.

For exercises **c**, **d**, **e**, **f**, *we assume* $2 < \lambda \leq 3$.

c (i). Prove that f_λ^2 is symmetric about the line$x = 1/2$; i.e.

$$f_\lambda^2\left(\frac{1}{[}2 - x\right) = f_\lambda^2\left(\frac{1}{2} + x\right).$$

(ii). Prove that f_λ^2 has a fixed point at c_λ, and that

$$(f_\lambda^2)'(c\lambda) = (f_\lambda'(c\lambda))^2.$$

(Hint: use the chain rule.)

***d** Prove that f_λ^2 takes its maximum value $\lambda/4$ at d_λ and $1-d_\lambda$, where

$$d_\lambda = \frac{1}{2} + \frac{1}{2}\sqrt{1-\frac{2}{\lambda}},$$

so that $f_\lambda(d_\lambda) = 1/2$. (Use the fact that the solutions of $(f_\lambda^2)'(x) = 0$ are $x = d_\lambda$, $x = 1-d_\lambda$ and $x = 1/2$.)

***e (i).** Prove that $c_\lambda < f_\lambda(\frac{1}{2}) < d_\lambda$, and deduce that $f_\lambda^2(\frac{1}{2}) > 1/2$.

(ii). Prove that f_λ^2 is increasing for $\frac{1}{2} \le x \le d_\lambda$.

f (i). Prove that

$$f_\lambda^2(x) - x = (f_\lambda(x) - x)g_\lambda(x),$$

where

$$g_\lambda(x) = \lambda^2 x^2 - (\lambda^2+\lambda)x + \lambda + 1.$$

(ii). Show that the solutions of $g_\lambda(x) = 0$ are

$$x = \frac{\lambda + 1 \pm \sqrt{(\lambda+1)(\lambda-3)}}{2\lambda},$$

and deduce that f_λ^2 has no fixed points in $(0,1)$, other than c_λ, if $2 < \lambda \le 3$.

Exercises for Section D

***a.** Prove Theorem 1.4 parts 2 and 3 from section D, p.24

Exercises **b** *and* **c** *deal with the function* $f(x) = \mu x - ax^3$ *for real constants* μ *abd* a *with* $a > 0$.

b. Show that $x = 0$ is a fixed point of f, and determine its stability for all values of μ and a.

c. Obtain the remaining fixed points of f. For which values of μ are there three fixed points of f, and for which values of μ is there only one fixed point of f? Determine the stability of the fixed points other than $x = 0$ for all values of μ and a.

d. Determine whether the fixed point $x = 0$ is semiasymptotically stable from the left or form the right in the following problems. (See question 17 from section D, page 29.)

(i). $f(x) = x^3 + \delta x^2 + x$, for δ a non-zero constant.

(ii) $f(x) = x + (ax^2 + bx)\cos x + (px^2 - b)\sin x$, for constants a, b, p with $a \neq 0$.

e. Show that between any two stable fixed points a and b of a continuous map f of an interval into itself, there must be a fixed point which is not stable. (Hint: use the Intermediate Value Theorem.)

Exercises for Section E

Let f be a continuous map of an interval I into itself. In what follows f^p denotes the p-fold composition $f \circ \cdots \circ f$ of f.

a.

(i) Define what it means for a point $x \in I$ to be periodic under f of period p.

(ii) Define what it means for x to be a stably periodic point of f.

(iii) Show that if x is a stably periodic point of f then so is $f(x)$.

(iv) Show that if x is periodic of period p then $(f^p)'(f^j(x)) = (f^p)'(x)$ for all $j \geq 0$.

b. Find all period 1 and period 2 points of $f(x) = x^2 - 1$ in $\mathbb{R}$. Show that the period 1 points are unstable and that the period two orbit is stable. (HInt: compute $f'(x)$ for each fixed point x and $(f^2)'(x)$ for one x in the single period two orbit.)

c. Let $f(x) = x^2 - 2$. Verify that f maps $[-2, 2]$ into $[-2, 2]$ and show that

$$f(2\cos(\theta)) = 2\cos(2\theta).$$

Hence find a formula for f^n for all n. Show that $2\cos\theta$ is fixed by f^p if and only if $2^p\theta = \pm\theta + 2k\pi$ for some integer k. Hence or otherwise write down all the points of periods two and three under f.

d. Define the Schwarzian derivative of a three-times differentiable real-valued function. Show that the Schwarzian derivative of any quadratic polynomial is < 0 or $= -\infty$ at all points.

e. State Singer's Theorem. Explain why this shows that$f$ as in **c** has at most three stable (also called attractive) periodic cycles. If possible explain why there are, in fact none.

Exercises for Section F

***a.** Let A be a 2×2 constant matrix and $\mathbf{v}$ a 2-dimensional consant column vector. Prove that if $A^n\mathbf{v}$ has a limiting direction, then this limit must be an eigenvector of A. (Hint: suppose $A^n\mathbf{v}/\|A^n\mathbf{v}\| \to \mathbf{w}$ for some $\mathbf{w} \neq \mathbf{0}$ as $n \to \infty$. Then consider $A(A^n\mathbf{v}/\|A^n\mathbf{v}\|$.)

b. Prove the following by mathematical induction.

(i). If $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ then $D^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}$.

(ii). If $J = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ then $J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$.

c. Prove the following by mathematical induction.

If $J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ the $J^n = |\lambda|^n \begin{pmatrix} \cos n\omega & \sin n\omega \\ -\sin n\omega & \cos n\omega \end{pmatrix}$, where (λ, ω) is the polar form of (α, β), that is, $(\alpha, \beta) = (\lambda\cos\omega, \lambda\sin\omega)$.

***d.** Show that

$A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$ is similar to $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, that is, $A = XBX^{-1}$ for an invertible 2×2 matrix X.

(Hint: show that the only eigenvalue of A is 2 and show that $(A-2)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{v}$ is an eigenvector of A with eigenvalue 2, and find the matrix of A with respect to the basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}$.

***e.** Show that $B = \begin{pmatrix} -1 & -2 \\ 4 & 3 \end{pmatrix}$ is similar to $B_2 = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$.

(Hint: show that the eigenvalues of both B and B_2 are $1 \pm 2i$, and find the corresponding eigenvectors, which will have complex coefficients in all cases.)

f. Consider the linear systems

(i) $X(n+1) = AX(n)$,
(ii) $Y(n+1) = BY(n)$,

where A and B are as in exercises **d** and **e**. Give the gerenal solutions and a fundamental set of solutions for each of **(i)** and **(ii)**. What happens to the general solution $X(n)$ of **(i)** as n gets large? What can you guess about the stability of the fixed point $X^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ of **(i)**?

Exercises for Section G

a. Suppose that f is a polynomial and that x_∞ is a zero of f with $f'(x_\infty) \neq 0$. Show that x_∞ is a stable fixed point of the Newton's method for F:

$$F(x) = x - \frac{f(x)}{f'(x)}.$$

(Hint: Show that $F'(x_\infty) = 0$.)

The next two questions concern Newton's method for $f(x) = x^2 - 2$.

b. Draw the graph of f. Now, using tangent lines to the graph of f, sketch the points $(x_0, f(x_0)$ and $(F(x_0), f(F(x_0))$ in each of the following cases

(i) $x_0 > 0$ and $f(x_0) < 0$
(ii) $x_0 > 0$ and $f(x_0) > 0$
(iii) $x_0 < 0$ and $f(x_0) > 0$
(iv) $x_0 < 0$ and $f(x_0) < 0$.

c. Verify the following by using the formula for $F(x_0)$ for this particular f. These should be apparent from the sketches. Define x_n inductively, given x_0, by

$$x_{n+1} = F(x_n) = x_n - \frac{x_n^2 - 2}{2x_n}.$$

Show by induction that, if $x_0 \neq 0$ and $x_0^2 \neq 2$, then:

(i) x_n has the same sign as x_0 for all $n \geq 0$.
(ii) $x_1^2 - 2 > 0$, and $x_n^2 > 2$ for all $n > 0$;
(iii) $0 < x_{n+1}^2 - 2 < x_n^2 - 2$ for all $n > 0$.

Hence or otherwise show that $\lim_{n\to\infty} x_n = \pm\sqrt{2}$, depending on whether $x_0 > 0$ or $x_0 < 0$.

The next few questions concern Newton's method for $f(x) = x^3 - 2$.

d. Sketch the graph of f and of the Newton's method $F(x) = x - \frac{x^3-2}{3x^2}$ for f.

***e.** Now let x_{n+1} be defined inductively by $x_{n+1} = F(x_n)$, if $x_n \neq 0$. Show that if $x_n \neq 0$ then:

$$f(x_{n+1}) = \frac{4(f(x_n))^2}{3(f'(x_n))^2}\left(4x_n + \frac{1}{x_n^2}\right) = \frac{4(f(x_n))^2}{27}\left(\frac{4}{x_n^3} + \frac{1}{x_n^6}\right)$$

(Hint: Remember that since f is a polynomial of degree 3, it is equal to its third order Taylor polynomial.)

f. Use induction, and the above, to show that, if $x_0 > 0$ and $f(x_0) \neq 0$:

(i) $x_n > 0$ for all $n \geq 0$;
(ii) $f(x_n) > 0$ for all $n \geq 1$;
(iii) $x_{n+1} < x_n$ for all $n \geq 1$;
(iv) $f(x_{n+1} \leq \frac{2}{3}f(x_n)$ for all $n \geq 1$.

Deduce that $\lim_{n\to\infty} x_n = 2^{1/3}$ if $x_0 > 0$.

Exercises for Section H

a. State the Intermediate Value Theorem.

b. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and suppose that there are $a < b$ such that $f(a) > b$ and $f(b) < a$. Prove that f has a fixed point in $[a, b]$.

***c.** Prove that if $f : [a, b] \to [a, b]$ is continuous and $f(a) = a$ and $f(b) = b$ but $f(x) \neq x$ for any x with $a < x < b$ then either $\lim_{n\to\infty} f^n(x) = a$ for all $x \in (a, b)$ or $\lim_{n\to\infty} f^n(x) = b$ for all $x \in (a, b)$. (Hint: Prove this by contradiction. Use the Intermediate Value Theorem.)

d. State Lemma 2.1 of [E]

The result of exercise **e** *is implied by Sarkovskiy's Theorem but the result of* **f** *is implied by the methods of proof of Sarkovskiy's Theorem, not by the statement of it .*

e. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and there are points $a < b < c$ with

$$f(a) = b, \quad f(b) = c, \quad f(c) = a.$$

Show that f has a point of period 2. (Hint: Use Lemma 2.1 of [E] to find an interval $[a_1, b_1] \subset [a, b]$ such that $f([a_1, b_1] \subset [b, c]$ and $f^2([a_1, b_1]) = [a, b]$.)

f. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and there are points $a < b < c < d$ with

$$f(a) = b, \quad f(b) = c, \quad f(c) = d, \quad f(d) = a.$$

Show that f has a point of period 3. (HInt: again, use Lemma 2.1 of [E].