41. The Mapping $w = \cos z$

In studying the geometric behavior of the trigonometric functions, we shall confine ourselves to the mapping $w = \cos z$, since the mapping $w = \sin z$ can be written in the form

$$w = -\cos\left(z + \frac{\pi}{2}\right),$$

and hence reduces to a shift

$$z_1 = z + \frac{\pi}{2}$$

of the z-plane in the direction of the positive real axis, followed first by the mapping $z_2 = \cos z_1$ and then by the mapping $w = -z_2$, corresponding to a rotation of the whole z_2-plane through the angle π about the origin.

First we consider the inverse images of the point w under the mapping $w = \cos z$, i.e., the roots of the equation

$$\cos z = w, \tag{9.49}$$

where w is an arbitrary finite complex number. Substituting (9.38) for $\cos z$ and writing

$$e^{iz} = t \tag{9.50}$$

for brevity, we find that t satisfies the equation

$$\frac{1}{2}\left(t + \frac{1}{t}\right) = w$$

or

$$t^2 - 2wt + 1 = 0, \tag{9.51}$$

with solutions

$$t_j = w + \sqrt{w^2 - 1} \qquad (j = 1, 2). \tag{9.52}$$

In (9.52) we do not write $\pm$ in front of the radical, since the square root is already understood to have two values (see Sec. 6). Obviously, the product of the two numbers t_1 and t_2 equals 1, and hence $t_1 \neq 0$, $t_2 \neq 0$. If we denote one of these numbers by τ and the other by $1/\tau$, (9.50) leads to two equations for determining z:

$$e^{iz} = \tau\,(\neq 0), \qquad e^{iz} = \frac{1}{\tau}\,(\neq 0).$$

According to Sec. 39, each of these equations has infinitely many solutions, found by using formula (9.25), i.e.,

$$iz' = \ln|\tau| + i \operatorname{Arg}\tau, \qquad iz'' = \ln\left|\frac{1}{\tau}\right| + i \operatorname{Arg}\frac{1}{\tau}$$

or

$$z' = \operatorname{Arg}\tau - i\ln|\tau|, \qquad z'' = -(\operatorname{Arg}\tau - i\ln|\tau|).$$

Therefore the solutions of (9.49) consist of two infinite sets of points z_n', z_n'' ($n = 0, \pm 1, \pm 2, \ldots$) lying on the lines $y = \pm\ln|\tau|$ parallel to the real axis, where each adjacent pair of points z_n' is separated by the distance 2π, and the same is true of the points z_n''. For each point z_n' on the line $y = -\ln|\tau|$ there is a point z_n'' on the line $y = \ln|\tau|$ which is symmetric to z_n' with respect to the origin (see Figure 9.8). For $w = \pm 1$ the roots τ and $1/\tau$ of equation (9.51) become ± 1, and then both lines coincide with the real axis, and both sets of points z_n', z_n'' also coincide.

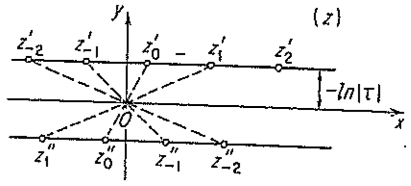

FIGURE 9.8

Thus, in any case, equation (9.49) has infinitely many solutions. It follows that 1) the function $w = \cos z$ maps the finite z-plane *onto* the finite w-plane, and 2) each point w has infinitely many inverse images in the z-plane. Moreover, the mapping is conformal at all points where

$$(\cos z)' = -\sin z \neq 0,$$

i.e., for

$$z \neq k\pi \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

Now consider the effect of the mapping $w = \cos z$ on straight lines parallel to one of the coordinate axes. Every line l parallel to the imaginary axis has an equation of the form

$$z = b + it \qquad (-\infty < t < \infty),$$

where b is a real number. The image of l under the mapping $w = \cos z$ is the curve L with equation

$$w = u + iv = \cos(b + it) = \cos b \cosh t - i \sin b \sinh t \tag{9.53}$$

[cf. (9.47)]. If $b = k\pi$, where k is an integer, (9.53) reduces to

$$w = \cos k\pi \cosh t = (-1)^k \cosh t,$$

i.e., w describes the ray $u \geqslant 1$, $v = 0$ twice if k is even, or the ray $u \leqslant -1$, $v = 0$ twice if k is odd (these rays are both parts of the real axis). On the other hand, if $b = (2k - 1)\pi/2$, (9.53) reduces to

$$w = (-1)^k i \sinh t,$$

i.e., w describes the whole imaginary axis once in the direction of increasing u if k is even, or once in the direction of decreasing u if k is odd. Finally, if b is not an integral multiple of $\pi/2$, we separate real and imaginary parts of (9.53), obtaining the following parametric equations for L:

$$u = \cos b \cosh t, \qquad v = -\sin b \sinh t. \tag{9.54}$$

Elimination of the parameter t leads to

$$\frac{u^2}{\cos^2 b} - \frac{v^2}{\sin^2 b} = 1, \tag{9.55}$$

where $\cos b \neq 0$, $\sin b \neq 0$. This is the equation of a hyperbola with semiaxes $|\cos b|$ and $|\sin b|$, and with foci at the points $w = \pm 1$.

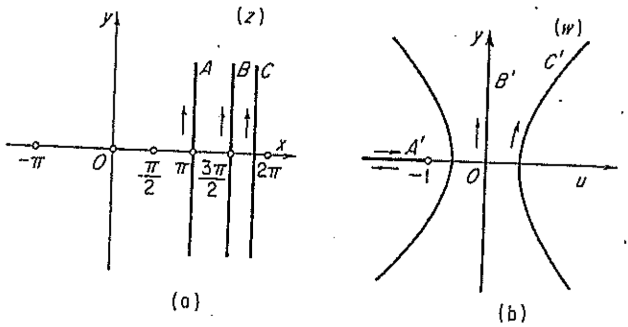

FIGURE 9.9

Remark. It should not be thought that the curve L with parametric representation (9.54) consists of both branches of the hyperbola with equation (9.55). In fact, it follows from (9.54) that u always has the same sign as $\cos b$, and only v changes sign, varying monotonically from $-\infty$ to $+\infty$ as t increases if $\sin b < 0$, or from $+\infty$ to $-\infty$ if $\sin b > 0$. In other words, L actually consists of just one of the branches of the hyperbola (9.55), i.e., the right-hand branch if $\cos b > 0$, or the left-hand branch if $\cos b < 0$. In every case, the mapping of the line l onto the curve L is one-to-one, and in fact, there is a one-to-one correspondence between the points of each of the half-lines into which l is divided by the real axis and the points of one of the half-branches into which L is divided by its vertex.

Example. Consider the three lines

$$A)\quad x = \pi,$$

$$B)\quad x = \frac{3\pi}{2},$$

$$C)\quad x = b \quad \left(\frac{3\pi}{2} < b < 2\pi\right),$$

shown in Figure 9.9(a). The images of these three lines are shown in Figure 9.9(b) and are A') a ray along the negative real axis, B') the imaginary axis itself, and C') the right-hand branch of a hyperbola.

Next suppose z describes a line λ parallel to the real axis, with equation

$$z = t + ic \qquad (-\infty < t < \infty),$$

where c is a real number. The image of λ under the mapping $w = \cos z$ is the curve Λ with equation

$$w = u + iv = \cos(t + ic) = \cos t \cosh c - i \sin t \sinh c. \tag{9.56}$$

If $c = 0$, λ is the real axis and Λ is the curve with equation

$$w = \cos t \qquad (-\infty < t < \infty),$$

i.e., w describes the segment $-1 \leqslant u \leqslant 1$ of the real-axis infinitely many times, in fact twice every time z goes a distance 2π along λ. If $c \neq 0$, we

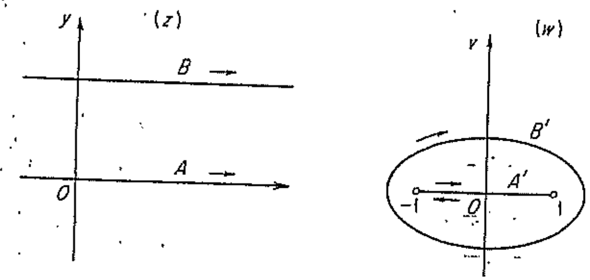

FIGURE 9.10

separate real and imaginary parts of (9.56), obtaining the following parametric equations for Λ:

$$u = \cosh c \cos t, \qquad v = -\sinh c \sin t. \tag{9.57}$$

Elimination of the parameter t gives

$$\frac{u^2}{\cosh^2 c} + \frac{v^2}{\sinh^2 c} = 1,$$

where $\cosh c \neq 0$, $\sinh c \neq 0$. This is the equation of an ellipse with semiaxes $|\cosh c|$ and $|\sinh c|$, and with foci at the points $w = \pm 1$. It follows from the representation (9.57) of the curve Λ that the point w describes the ellipse infinitely many times in the same direction, where each circuit around the ellipse corresponds to a displacement of the point z a distance 2π along the line λ. The two lines

$$A)\quad y = 0,$$

$$B)\quad y = c \neq 0,$$

and their images A' and B' in the w-plane are shown in Figure 9.10.

To summarize, *the mapping* $w = \cos z$ *transforms the two one-parameter families of straight lines parallel to the coordinate axes, but distinct from the lines* $x = k\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$ *or* $y = 0$, *into a one-parameter family of confocal hyperbolas and a one-parameter family of confocal ellipses, with common foci at the points* $w = \pm 1$. Moreover, since the mapping is conformal at all points of the z-plane except at the points

$$z = k\pi \qquad (k = 0, \pm 1, \pm 2, \ldots), \tag{9.58}$$

whose images are precisely the foci $w = \pm 1$, and since the two families of lines in the z-plane form an orthogonal system (see p. 134), it follows that the family of hyperbolas and the family of ellipses in the w-plane also form an orthogonal system.

42. The Image of a Half-Strip under $w = \cos z$

We now pose the problem of finding a domain G in the z-plane on which the function $w = \cos z$ is one-to-one. The choice of such a domain can be made in many ways. It is only necessary to make sure that G does not contain two or more inverse images of the same point w. For example,

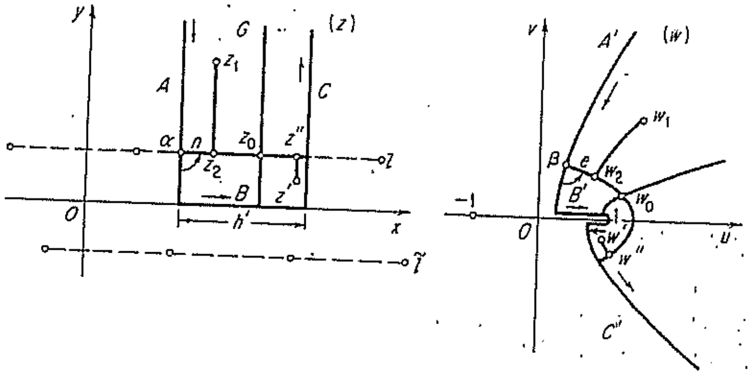

FIGURE 9.11

suppose G is the open half-strip of width h $(0 < h \leqslant 2\pi)$ whose sides are parallel to the imaginary axis and whose base is a segment of the real axis [see Figure 9.11(a)]. Then $w = \cos z$ is a one-to-one function on G. In fact, as we know from the preceding section, if $z_0 \in G$ and $w_0 = \cos z_0$, all the other inverse images of w_0 must lie either on the line l parallel to the real axis and passing through the point z_0, or on the line $\bar{l}$ which is symmetric to l with respect to the real axis. But the distance from z_0 to any other inverse image of w_0 lying on l is an integral multiple of 2π, and since the width of the half-strip does not exceed 2π, none of the other inverse images on l can belong to either G or its boundary. Moreover, G and the line $\bar{l}$ obviously

have no points in common. It follows that the function $w = \cos z$ is a one-to-one conformal mapping of G onto some set of points $\mathscr{G}$ in the w-plane [note that none of the points (9.58) belongs to G].

To construct $\mathscr{G}$, we proceed as follows: Let the point $z = x + iy$ trace out the boundary γ of the half-strip in the way indicated in Figure 9.11(a), i.e., let z consecutively traverse the following three curves in the z-plane:

A) The left-hand side of the half-strip, in the direction of decreasing y;
B) The base of the half-strip, in the direction of increasing x;
C) The right-hand side of the half-strip, in the direction of increasing y.

This causes the image point $w = u + iv = \cos z$ to consecutively traverse the following three curves in the w-plane, as indicated in Figure 9.11(b):[11]

A') Half of a branch of a hyperbola, in the direction of decreasing v;
B') Part of the segment $-1 \leqslant u \leqslant 1$ of the real axis, first in the direction of increasing u, and then in the direction of decreasing u;
C') Half of the branch of another hyperbola, in the direction of decreasing v.

The union of the three curves A', B' and C', which we denote by Γ, is the image (under $w = \cos z$) of γ, the boundary of G. Clearly, Γ is a closed Jordan curve, passing through the point at infinity. Therefore, Γ divides the extended w-plane into two disjoint domains D_1 and D_2, with Γ as their common boundary (cf. Secs. 18, 22). According to Theorem 6.3, one of these two domains is the set $\mathscr{G}$, the image of G. To ascertain whether $D_1 = \mathscr{G}$ or $D_2 = \mathscr{G}$, we choose a point $z_0 \in G$ and find its image $w_0 = \cos z_0$. Clearly $w_0 \notin \Gamma$, since otherwise one of the inverse images of w_0 would belong to G and another to γ, which, as we have just seen, is impossible. Therefore, w_0 belongs to either D_1 or D_2, and the domain containing w_0 is $\mathscr{G}$ (see the remark on p. 100).

Remark 1. In the case of an elementary mapping like $w = \cos z$, we can use the detailed geometry of the mapping to prove that $\mathscr{G}$ coincides with one of the domains D_1 and D_2 into which Γ divides the w-plane, thereby avoiding the topological considerations of Chap. 6. Let z_0 be an arbitrary (but fixed) point of G, and let $w_0 = \cos z_0$ be its image. As already noted, one of the domains D_1 and D_2 contains w_0; let this domain be denoted simply by D. First we show that the image w_1 of any point $z_1 \in G$ different from z_0 belongs to D, so that $\mathscr{G} \subset D$. Through the point z_0 draw the line l parallel to the real axis, and through the point z_1 draw the line parallel to the imaginary axis.

[11] Of course, the specific appearance of the curves A', B' and C', and the directions in which they are traversed, depends on the location and width of the half-strip in the z-plane. In this respect, Figure 9.11 represents only one possible case (see Remark 3, p. 157 and Problem 9.16). For example, if the half-strip is of width 2π, the curve B' will consist of the whole segment $-1 \leqslant u \leqslant 1$ traced out *twice*.

Let z_2 be the point of intersection of these two lines [see Figure 9.11(a)]. Then the polygonal curve[12]

$$\lambda = \overline{z_0z_2} + \overline{z_2z_1}$$

joins z_0 to z_1, and moreover $\lambda \subset G$. (Either $z_0 = z_2$ or $z_2 = z_1$ is possible, but not both.) The image of λ under the mapping $w = \cos z$ is a curve

$$\Lambda = \widehat{w_0w_2} + \widehat{w_2w_1},$$

where $\widehat{w_0w_2}$ is an arc of an ellipse and $\widehat{w_2w_1}$ is an arc of a hyperbola, both with foci ± 1 ($w_1 = \cos z_1$, $w_2 = \cos z_2$). Clearly $\Lambda \cap \Gamma = 0$, since $\lambda \cap \gamma = 0$ and no pair of points, one in G and the other on its boundary, can have the same image. Therefore $\Lambda \subset D$, and hence $w_1 \in D$, since Λ connects w_0 to w_1, i.e., $\mathscr{G} \subset D$, as asserted.

Next, let w' be a point of D different from w_0. We now show that w' has a (unique) inverse image belonging to G, so that $\mathscr{G} \supset D$. Since $w' \notin \Gamma$, there is a unique ellipse with foci ± 1 passing through w_0, and a unique hyperbola with foci ± 1 passing through w'. Let w'' be the point of intersection of these two curves [see Figure 9.11(b)]. Then the curve

$$\Lambda = \widehat{w_0w''} + \widehat{w''w'}$$

joins w_0 to w', and moreover $\Lambda \subset D$. (Either $w_0 = w''$ or $w'' = w'$ is possible, but not both.) Any inverse image of Λ under $w = \cos z$ is a polygonal curve

$$\lambda = \overline{z_0z''} + \overline{z''z'},$$

where $\overline{z_0z''}$ is a segment of a line parallel to the real axis, $\overline{z''z'}$ is a segment of a line parallel to the imaginary axis, and z', z'' are any two points such that $w' = \cos z'$, $w'' = \cos z''$. Obviously $\lambda \cap \gamma = 0$, since $\Lambda \cap \Gamma = 0$. Therefore $\lambda \subset G$, and hence $z' \in G$, since λ connects z_0 to z', i.e., $\mathscr{G} \subset D$, as asserted, and the proof is complete.[13]

Remark 2. We now indicate another way of deciding which of the two domains D_1 and D_2 with boundary Γ coincides with $\mathscr{G}$. This method does not require knowing the image w_0 of a point $z_0 \in G$, and goes as follows: As

[12] By $\widehat{ab}$ we mean a curve joining a to b (the curve in question is always apparent from the context). In the case where the curve is the line segment joining a to b, we write $\overline{ab}$ instead of $\widehat{ab}$. If γ and δ are two curves such that the final point of γ coincides with the initial point of δ, then by $\gamma + \delta$ we mean the *oriented* curve obtained by first going along γ from its initial point to its final point and then along δ from its initial point to its final point. We use the symbol + instead of $\cup$ to emphasize that $\gamma + \delta$ is traversed in the direction indicated. It should be noted that $\widehat{z_0z_2} \cup \widehat{z_2z_1} = \widehat{z_2z_1} \cup \widehat{z_0z_2}$, but $\widehat{z_2z_1} + \widehat{z_0z_2}$ is meaningless.

[13] We can now state that z', z'' and hence $\overline{z_0z''}$, $\overline{z''z'}$ are unique, since $z' \in G$, $z'' \in G$.

a moving point z traces out the boundary γ of the domain G in a certain direction, the image point w traces out the boundary Γ of the domain $\mathscr{G}$ in a corresponding direction. For example, the direction along γ indicated by the arrows in Figure 9.11(a) corresponds to the direction along Γ indicated by the arrows in Figure 9.11(b), and if one of these directions were reversed, the other would also have to be reversed. Suppose an observer (labeled *I*) follows the moving point z along γ, and suppose another observer (labeled *II*) follows the moving image point $w = \cos z$ along Γ. Then if *I* finds G on his left, *II* finds $\mathscr{G}$ on his left, while if *I* finds G on his right, *II* finds $\mathscr{G}$ on his right. To prove this, suppose *I* moves along the side A of the half-strip G in the direction shown in Figure 9.11(a) and hence finds G on his left. From some point $\alpha \in A$ ($\operatorname{Im} \alpha > 0$) we draw a segment n of the normal to A pointing into G, i.e., a certain line segment parallel to the real axis. Obviously $\mathscr{G}$ must contain the image of this segment, which is a certain elliptical arc e drawn from the point $\beta = \cos \alpha$ belonging to A', as shown in Figure 9.11(b). In other words, $\mathscr{G}$ is uniquely characterized as the domain with boundary Γ into which e points. The assertion that *I* finds G on his left is equivalent to the assertion that in order to enter G along the segment n at the point α, *I* must make a "left turn," i.e., a counterclockwise rotation through 90°. But $w = \cos z$ has a nonzero derivative at α, and hence is a conformal mapping of the first kind which preserves not only angles but also the directions in which they are measured, as discussed in Sec. 31. Therefore, in order to enter $\mathscr{G}$ along the arc e at the point β, *II* must also make a left turn, i.e., *II* finds $\mathscr{G}$ on his left, as required.

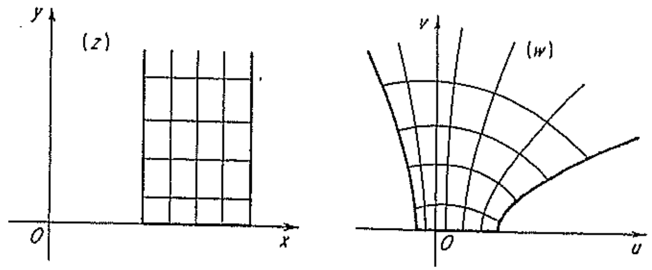

FIGURE 9.12

Remark 3. Once again (cf. footnote 11), we mention that the specific appearance of the domain $\mathscr{G}$ depends on the location and width of the half-strip G. The case where the base of G is contained in an interval of the form $[k\pi, (k+1)\pi]$ is shown in Figure 9.12. The situation illustrated by Figure 9.11 corresponds to the case where the base of G is an interval containing a point of the form $k\pi$ in its interior. In this case, the boundary Γ is "folded over" at either $w = -1$ or $w = +1$ (or both).