16. Prove that a Morse-Smale diffeomorphism of $[0,1]$ is structurally stable.

17. Prove that the map $f(x) = x^3 + \frac{3}{4}x$ is a Morse-Smale diffeomorphism on the interval $[-\frac{1}{2}, \frac{1}{2}]$.

§1.10 SARKOVSKII'S THEOREM

In this section, we will prove a remarkable theorem due to Sarkovskii. The theorem only holds for maps of the real line, but nevertheless is amazing for its lack of hypotheses (f is only assumed continuous) and strong conclusion. We caution the reader that, as this is our first major theorem, the material in this section is a little "heavier" than in previous sections. As a warmup, and also as a means of highlighting the importance of period three points, we will prove a special case.

Theorem 10.1. *Let $f: \mathbf{R} \to \mathbf{R}$ be continuous. Suppose f has a periodic point of period three. Then f has periodic points of all other periods.*

Proof. The proof will depend on two elementary observations. First, if I and J are closed intervals with $I \subset J$ and $f(I) \supset J$, then f has a fixed point in I. This is, of course, a simple consequence of the Intermediate Value Theorem. See Fig. 10.1.

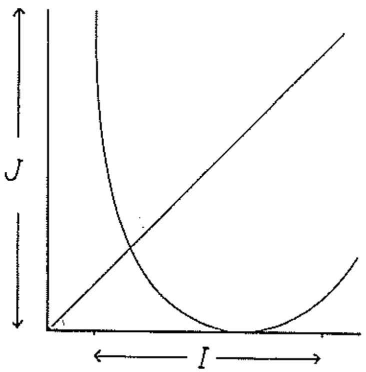


Fig. 10.1

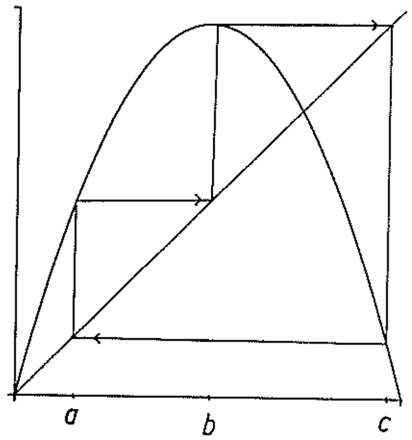


Fig. 10.2. The map $F_{3.839}(x) = 3.839x(1-x)$.

The second observation is the following: suppose $A_0, A_1, \ldots, A_n$ are closed intervals and $f(A_i) \supset A_{i+1}$ for $i = 0, \ldots, n-1$. Then there exists at least one subinterval J_0 of A_0 which is mapped onto A_1. There is a similar subinterval in A_1 which is mapped onto A_2, and thus there is a subinterval $J_1 \subset J_0$ having the property that $f(J_1) \subset A_1$ and $f^2(J_1) = A_2$. Continuing in this fashion, we find a nested sequence of intervals which map into the various A_i in order. Thus there exists a point $x \in A_0$ such that $f^i(x) \in A_i$ for each i. We say that $f(A_i)$ *covers* A_{i+1}. See Exercise 1.

To prove the Theorem, let $a, b, c \in \mathbf{R}$ and suppose $f(a) = b$, $f(b) = c$, and $f(c) = a$. We assume that $a < b < c$. The only other possibility, $f(a) = c$, is handled similarly. This situation arises in the quadratic map F_μ for sufficiently large μ, and even for some $\mu < 4$. In fact, we will exploit this fact later when we discuss the case $\mu = 3.839$ in detail in §1.13. See Fig. 10.2.

Let $I_0 = [a,b]$ and $I_1 = [b,c]$ and note that our assumptions imply $f(I_0) \supset I_1$ and $f(I_1) \supset I_0 \cup I_1$. The graph of f shows that there must be a fixed point for f between b and c. Similarly, f^2 must have fixed points between a and b, and it is easy to see that at least one of these points must have period two. So we let $n \geq 2$; our goal then is to produce a periodic point of prime period $n > 3$.

Inductively, we define a nested sequence of intervals $A_0, A_1, \ldots, A_{n-2} \subset I_1$ as follows. Set $A_0 = I_1$. Since $f(I_1) \supset I_1$, there is a subinterval $A_1 \subset A_0$ such that $f(A_1) = A_0 = I_1$. Then there is a subinterval $A_2 \subset A_1$ such that $f(A_2) = A_1$, so that $f^2(A_2) = A_0 = I_1$. Continuing, we find a subinterval $A_{n-2} \subset A_{n-3}$ such that $f(A_{n-2}) = A_{n-3}$. According to our second observation above, if $x \in A_{n-2}$, then $f(x), f^2(x), \ldots, f^{n-2}(x) \subset A_0$ and, indeed, $f^{n-2}(A_{n-2}) = A_0 = I_1$.

Now since $f(I_1) \supset I_0$, there exists a subinterval $A_{n-1} \subset A_{n-2}$ such that $f^{n-1}(A_{n-1}) = I_0$. Finally, since $f(I_0) \supset I_1$ we have, $f^n(A_{n-1}) \supset I_1$ so that $f^n(A_{n-1})$ covers A_{n-1}. It follows from our first observations that f^n has a fixed point p in A_{n-1}.

We claim that p actually has prime period n. Indeed, the first $n-2$ iterations of p lie in I_1, the $(n-1)^{st}$ lies in I_0, and the n^{th} is p again. If $f^{n-1}(p)$ lies in the interior of I_0 then it follows easily that p has prime period n. If $f^{n-1}(p)$ happens to lie on the boundary, then $n = 2$ or 3, and again we are done.

q.e.d.

This theorem is just the beginning of the story. Sarkovskii's Theorem gives a complete accounting of which periods imply which other periods for continuous maps of $\mathbf{R}$. Consider the following ordering of the natural numbers:

$$3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2 \cdot 3 \rhd 2 \cdot 5 \rhd \cdots \rhd 2^2 \cdot 3 \rhd 2^2 \cdot 5 \rhd \cdots$$

$$\rhd 2^3 \cdot 3 \rhd 2^3 \cdot 5 \rhd \cdots\cdots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1.$$

That is, first list all odd numbers except one, followed by 2 times the odds, 2^2 times the odds, 2^3 times the odds, etc. This exhausts all the natural numbers with the exception of the powers of two which we list last, in decreasing order. This is the Sarkovskii ordering of the natural numbers. Sarkovskii's Theorem is:

Theorem 10.2. *Suppose $f: \mathbf{R} \to \mathbf{R}$ is continuous. Suppose f has a periodic point of prime period k. If $k \rhd \ell$ in the above ordering, then f also has a periodic point of period ℓ.*

Before proving this Theorem, we note several consequences.

Remarks.

1. If f has a periodic point whose period is not a power of two, then f necessarily has infinitely many period points. Conversely, if f has only finitely many periodic points, then they all necessarily have periods which are powers of two. This fact will reappear when we discuss the period-doubling route to chaos in a later section.

2. Period 3 is the greatest period in the Sarkovskii ordering and therefore implies the existence of all other periods, as we saw above.

3. The converse of Sarkovskii's Theorem is also true! There are maps which have periodic points of period p and no "higher" period points according to the Sarkovskii ordering. We give several examples of this at the end of this section.

We will give an elementary proof of Sarkovskii's Theorem due to Block, Guckenheimer, Misiurewicz and Young. The proof rests mainly on the two observations which we used above. For two closed intervals, I_1 and I_2, we will introduce the notation $I_1 \to I_2$ if $f(I_1)$ covers I_2. If we find a sequence of intervals $I_1 \to I_2 \to \ldots \to I_n \to I_1$, then our previous observations show that there is a fixed point of f^n in I_1.

We first assume that f has a periodic point x of period n with n odd and $n > 1$. Suppose that f has no periodic points of odd period less than n. Let $x_1, \ldots, x_n$ be the points on the orbit of x, enumerated from left to right. Note that f permutes the x_i. Clearly, $f(x_n) < x_n$. Let us choose the largest i for which $f(x_i) > x_i$. Let I_1 be the interval $[x_i, x_{i+1}]$. Since $f(x_{i+1}) < x_{i+1}$, it follows that $f(x_{i+1}) \le x_i$ and so we have that $f(I_1) \supset I_1$. Therefore, $I_1 \to I_1$.

Since x does not have period 2, it cannot be that $f(x_{i+1}) = x_i$ and $f(x_i) = x_{i+1}$ so that $f(I_1)$ contains at least one other interval of the form $[x_j, x_{j+1}]$. A priori, there may be several such intervals, but we will see below that in fact there is only one. Let $\mathcal{O}_2$ denote the union of intervals of the form $[x_j, x_{j+1}]$ that are covered by $f(I_1)$. Hence we have $\mathcal{O}_2 \supset I_1$ but $\mathcal{O}_2 \neq I_1$, and if I_2 is any interval in $\mathcal{O}_2$ of the form $[x_j, x_{j+1}]$, then $I_1 \to I_2$.

Now let $\mathcal{O}_3$ denote the union of intervals of the form $[x_j, x_{j+1}]$ that have the property that they are covered by the image of some interval in $\mathcal{O}_2$. Continuing inductively, we let $\mathcal{O}_{\ell+1}$ be the union of intervals that are covered by the image of some interval in $\mathcal{O}_\ell$. Note that, if $I_{\ell+1}$ is any interval in $\mathcal{O}_{\ell+1}$, there is a collection of intervals $I_2, \ldots, I_\ell$ with $I_j \subset \mathcal{O}_j$ which satisfy $I_1 \to I_2 \to \ldots \to I_\ell \to I_{\ell+1}$.

Now the $\mathcal{O}_\ell$ form an increasing union of intervals. Since there are only finitely many x_j, it follows that there is an ℓ for which $\mathcal{O}_{\ell+1} = \mathcal{O}_\ell$. For this ℓ we must have that $\mathcal{O}_\ell$ contains all intervals of the form $[x_j, x_{j+1}]$, for otherwise x would have period less than n.

We claim that there is at least one interval $[x_j, x_{j+1}]$ different from I_1 in some $\mathcal{O}_k$ whose image covers I_1. This follows since there are more x_i's on one side of I_1 than on the other (n is odd.) Hence some x_i's must change sides under the action of f, and some must not. Consequently, there is at least one interval whose image covers I_1.

Now let us consider chains of intervals $I_1 \to I_2 \to \ldots \to I_k \to I_1$ where each I_ℓ is of the form $[x_j, x_{j+1}]$ for some j and where $I_2 \neq I_1$. We do not assume that $I_\ell \subset \mathcal{O}_\ell$. By the above observations, there is at least one such chain. Let us choose the smallest k for which this happens, i.e., $I_1 \to I_2 \to \dots \to I_k \to I_1$ is the shortest path from I_1 to I_1 except, of course, $I_1 \to I_1$. We therefore find a diagram as in Fig. 10.3.

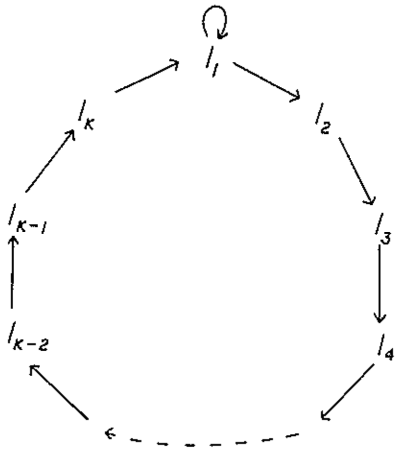


Fig. 10.3.

Now, if $k < n-1$, then one of the loops $I_1 \to I_2 \to \dots \to I_k \to I_1$ or $I_1 \to \dots \to I_k \to I_1 \to I_1$ gives a fixed point of f^m with m odd and $m < k$. This point must have prime period $< k$ since $I_1 \cap I_2$ consists of only one point, and that point has period $> m$. Therefore $k = n-1$.

Since k is the smallest integer that works, we cannot have $I_\ell \to I_j$ for any $j > \ell + 1$. It follows that the orbit of x must be ordered in $\mathbf{R}$ in one of two possible ways, as depicted in Fig. 10.4.

It follows that we can extend the diagram depicted in Fig. 10.3 to that shown in Fig. 10.5. Sarkovskii's Theorem for the special case of n odd is now immediate. Periods larger than n are given by cycles of the form $I_1 \to \dots \to I_{n-1} \to I_1 \to \dots I_1$. The smaller even periods are given by cycles of the form

$$I_{n-1} \to I_{n-2} \to I_{n-1},$$

$$I_{n-1} \to I_{n-4} \to I_{n-3} \to I_{n-2} \to I_{n-1}$$

and so forth. For the case of n even, we first note that f must have a periodic point of period 2. This follows from the above arguments provided we can

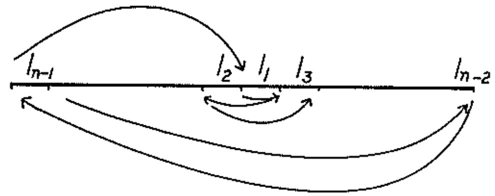


Fig. 10.4. One possible ordering of the I_j.
The other is the mirror image.

guarantee that some x_i's change sides under f and some do not (use the facts that $I_{n-1} \leftarrow I_{n-2}$ and $I_{n-1} \to I_{n-2}$). If this is not the case, then all of the x_i's must change sides and so $f[x_1, x_i] \supset [x_{i+1}, x_n]$ and $f[x_{i+1}, x_N] \supset [x_1, x_i]$. But then, our observation above produces a period 2 point in $[x_1, x_i]$.

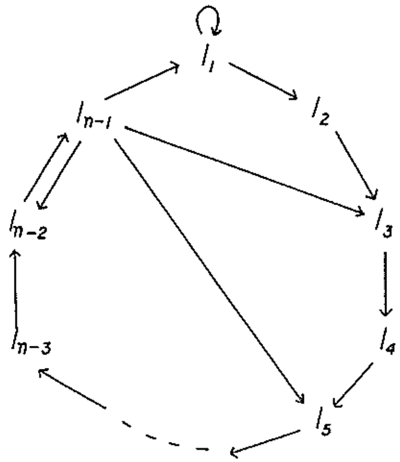


Fig. 10.5.

The Theorem now will be proved for $n = 2^m$ as follows. Let $k = 2^\ell$ with $\ell < m$. Consider $g = f^{k/2}$. By assumption, g has a periodic point of period $2^{m-\ell+1}$. Therefore, g has a point which has period 2. This point has period 2^ℓ for f. The final case is now $n = p \cdot 2^m$ where p is odd. This case can be reduced to the previous two. We leave these reductions as Exercises.

q.e.d.

We now turn to the converse of Sarkovskii's Theorem. To produce a map with period 5 and no period 3, consider a map $f:[1,5] \to [1,5]$ which satisfies

$$f(1) = 3$$
$$f(3) = 4$$
$$f(4) = 2$$
$$f(2) = 5$$
$$f(5) = 1$$

so that 1 is periodic of period 5. Suppose that f is linear between these integers, i.e., the graph is as shown in Fig. 10.6.

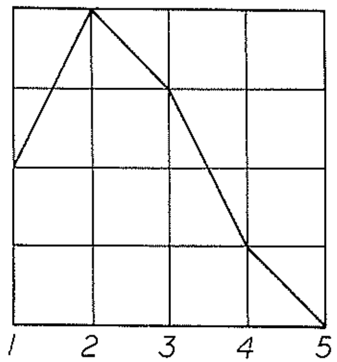


Fig. 10.6.

It is easy to check that

$$f^3[1,2] = [2,5]$$
$$f^3[2,3] = [3,5]$$
$$f^3[4,5] = [1,4]$$

so f^3 has no fixed points in any of these intervals. It is true that $f^3[3,4] = [1,5]$ so that f^3 has at least one fixed point in $[3,4]$. But we claim that this point is unique, and therefore must be the fixed point for f, not the period 3 point. Indeed, $f:[3,4] \to [2,4]$ is monotonically decreasing, as is $f:[2,4] \to [2,5]$ and $f:[2,5] \to [1,5]$. Therefore f^3 is monotonically decreasing on $[3,4]$ and the fixed point is unique.

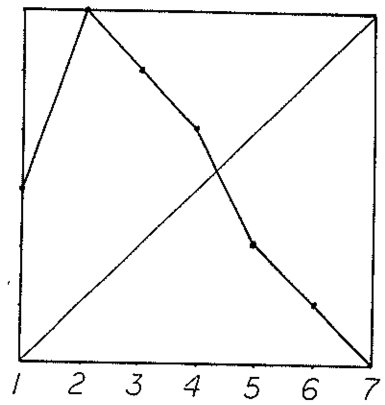
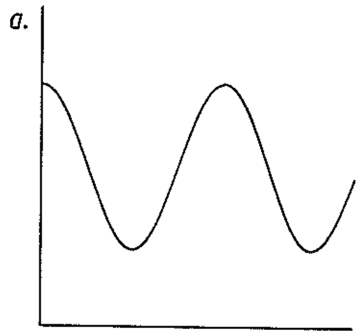


Fig. 10.7.

The graph, shown in Fig. 10.7, produces period 7 but not period 5.

This process is easily generalized to give the first portion of the Sarkovskii ordering. For the even periods, we will introduce a trick. Let $f: I \to I$ be continuous. We will construct a new function F, the double of f, whose periodic points will have exactly twice the period of those of f, plus one additional fixed point. The procedure for producing F is as follows. Divide the interval I into thirds. Compress the graph of f into the upper left corner of $I \times I$ as shown on Fig. 10.8.a. The rest of the graph is filled in as in Fig. 10.8.b.



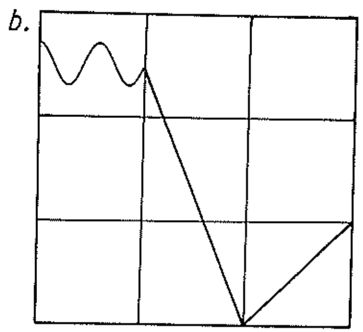


Fig. 10.8. Fig. 10.8.a. gives the graph of $f(x)$ while Fig. 10.8.b. gives the graph of its double, $F(x)$.

The map F is piecewise linear on $[1/3, 2/3]$ and $[2/3, 1]$. Moreover, $F(\frac{2}{3}) = 0$, $F(1) = \frac{1}{3}$, and F is continuous.

Note that F maps $[0, \frac{1}{3}]$ into $[\frac{2}{3}, 1]$ and vice versa. Also note that if $x \in [\frac{1}{3}, \frac{2}{3}]$ and x is not the fixed point, then there exists n so that $F^n(x) \in [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. This implies that there are no other F-periodic points in $(\frac{1}{3}, \frac{2}{3})$. Exercise 7 shows that if x is a periodic point of period n for f, then $x/3$ is periodic of period $2n$ for F. On the other hand, if y is F-periodic then either y or $F(y)$ lies in $[0, \frac{1}{3}]$ and Exercise 9 shows that $3y$ or $3F(y)$ is f-periodic. Thus to produce a map with period 10 but not period 6, we need only double the graph of a function with period 5 but not period 3.

As a final remark, we must emphasize that Sarkovskii's Theorem is very definitely only a one-dimensional result. There is no higher dimensional analogue of this result. In fact, the Theorem does not even hold on the circle. For example, the map which rotates all points on the circle by 120° makes all points periodic with period three. There are no other periods whatsoever.

Exercises

1. Suppose $A_0, A_1, \ldots, A_n$ are closed intervals and $f(A_i) \supset A_{i+1}$ for $i = 0, \ldots, n-1$. Prove that there exists a point $x \in A_0$ such that $f^i(x) \in A_i$ for each i.

2. Prove that if f has period $p \cdot 2^m$ with p odd, then f has period $q \cdot 2^m$ with q odd, $q > p$.

3. Prove that if f has period $p \cdot 2^m$ with p odd, then f has period 2^ℓ, $\ell \le m$.

4. Prove that if f has period $p \cdot 2^m$ with p odd, then f has period $q \cdot 2^m$ with q even.

5. Construct a piecewise linear map with period $2n+1$.

6. Give a formula for $F(x)$ in terms of $f(x)$, where $F(x)$ is the double of $f(x)$.

7. Prove that $F(x)$, the double of $f(x)$, has a periodic point of period $2n$ at $x/3$ iff x has f-period n.

8. Construct a map that has periodic points of period 2^j for $j < \ell$ but not period 2^ℓ.

9. Prove that if $F(x)$, the double of $f(x)$, has a periodic point p that is not fixed, then either p or $F(p)$ lies in $[0, \frac{1}{3}]$. Prove that, in this case, either $3p$ or $3F(p)$ is a periodic point for f.

§1.11 THE SCHWARZIAN DERIVATIVE

In this section, we describe a tool first introduced into the study of one-dimensional dynamical systems by Singer in 1978. This is the Schwarzian derivative. Actually, the Schwarzian derivative plays an important role in complex analysis, where it is used as a criterion for a complex function to be a linear fractional transformation. In one-dimensional dynamics, the Schwarzian derivative is a valuable tool for a number of reasons. In this section, we will show how it may be used to establish an upper bound on the number of attracting periodic orbits that certain maps may have. We will also use it to prove that other maps have an entire interval on which the map is chaotic. Later, in §§ 17–19, the Schwarzian derivative will play an important role in our discussion of how families of maps like the quadratic family make the transition from simple to chaotic dynamics.

Definition 11.1 The Schwarzian derivative of a function f at x is

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

For example, if $F_\mu(x) = \mu x(1-x)$ is our quadratic model mapping, then $SF_\mu(x) = -6/(1-2x)^2$, so that $SF_\mu(x) < 0$ for all x (even $x = 1/2$, the critical point, at which $SF_\mu(x) = -\infty$).

For us, functions with negative Schwarzian derivative will be most important. Besides the quadratic map, many other functions have negative Schwarzian derivatives. For example, $S(e^x) = -1/2$ and $S(\sin x) = -1 - \frac{3}{2}(\tan^2 x) < 0$. Many polynomials have this property, as the following proposition shows.

Proposition 11.2. *Let $P(x)$ be a polynomial. If all of the roots of $P'(x)$ are real and distinct, then $SP < 0$.*

Proof. Suppose

$$P'(x) = \prod_{i=1}^{N} (x - a_i)$$