Chapter 6

Series

Adding up (infinitely many) different things: e.g. Maclaurin series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Sometimes this makes sense (the series *converges*): sometimes it doesn't (the series *diverges*).

6.1 Convergence and Divergence (7.6.1)

Recall the notation

$$\sum_{r=0}^{n} a_r = a_0 + a_1 + a_2 + \cdots + a_n,$$

$$\sum_{r=0}^{\infty} a_r = a_0 + a_1 + a_2 + \cdots$$

Examples

$$\sum_{r=0}^{3} r^2 = 0^2 + 1^2 + 2^2 + 3^2 = 14.$$

$$\sum_{r=1}^{5} \frac{1}{r} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} (= \frac{137}{60}).$$

$$\sum_{r=0}^{\infty} \frac{x^r}{r!} = e^x.$$

$$\sum_{r=1}^{\infty}(-1)^{r+1}\frac{x^r}{r} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \ln(1+x).$$

(Note often see $(-1)^r$ or $(-1)^{r+1}$ in series. $(-1)^r$ is $+1$ when r is even, -1 when r is odd. $(-1)^{r+1}$ is the other way round. These give us *alternating* series.)

Finite series always make sense, but infinite ones may or may not.

Given an infinite series $\sum_{r=0}^{\infty} a_r$, define its *partial sums* S_n by cutting it off after a_n

$$S_n = \sum_{r=0}^{n} a_r$$

(these all make sense).

Say that the series *converges* if the partial sums get closer and closer to some *finite* value L, i.e. if $S_n \to L$ as $n \to \infty$. We write

$$\sum_{r=0}^{\infty} a_n = L.$$

We say that the series *diverges* otherwise.

Examples

a) Consider

$$\sum_{r=0}^{\infty}\frac{1}{2^r} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

The partial sums are 1, 3/2, 7/4, 15/8, etc., which clearly get closer and closer to 2. Thus the series is *convergent*, and

$$\sum_{r=0}^{\infty}\frac{1}{2^r} = 2.$$

b) Consider

$$\sum_{r=0}^{\infty}(-1)^r = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

The partial sums are 1, 0, 1, 0, 1, 0, etc., which clearly don't approach any particular value. Thus the series is *divergent*.

These examples illustrate an important fact: if the terms a_r don't get closer and closer to zero, then $\sum_{r=0}^{\infty} a_r$ must diverge.

However (equally important), the opposite is not true. Just because the terms get closer and closer to zero, it doesn't mean the series must converge. For example

$$\sum_{r=1}^{\infty} \frac{1}{r} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

diverges.

6.2 Geometric series (7.3.2, 7.6.1)

One of the few examples when we can actually calculate the value of an infinite series.

A geometric series is one in which each term is a multiple of the previous one, i.e.

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots$$

a is called the *first term* and r is the *common ratio.*

The partial sums are given by

$$S_n = a + ar + ar^2 + \cdots + ar^n.$$

We can work these out with a trick:

$$rS_n = ar + ar^2 + \cdots + ar^n + ar^{n+1},$$

so

$$S_n - rS_n = a - ar^{n+1},$$

or

$$S_n(1-r) = a(1 - r^{n+1}),$$

or

$$S_n = a\left(\frac{1 - r^{n+1}}{1-r}\right).$$

What happens as $n \to \infty$. If $-1 < r < 1$ then $r^{n+1} \to 0$, and $S_n \to \frac{a}{1-r}$. If $r \leq -1$ or $r \geq 1$, then the terms aren't getting smaller, and the series diverges.

A geometric series $\sum_{r=0} ar^n$ *converges to* $\frac{a}{1-r}$ *if* $-1 < r < 1$*, and diverges otherwise.*

6.3 Convergence tests (7.6.2, 7.6.3)

In most examples, it is impossible to work out the partial sums S_n, and we have to make do with deciding whether the series converges or diverges. There are a number of tests which help to do this.

The comparison test

If $\sum_{r=0}^{\infty} a_r$ converges, and $0 \leq |b_r| \leq a_r$ for all r, then $\sum_{r=0}^{\infty} b_r$ also converges.
If $\sum_{r=0}^{\infty} a_r$ diverges, and $0 \leq a_r \leq b_r$ for all r, then $\sum_{r=0}^{\infty} b_r$ also diverges.

Intuitively obvious.

Examples

a) Consider the factorial series

$$\sum_{r=0}^{\infty} \frac{1}{r!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Each term is less than or equal to the corresponding term in

$$1 + \sum_{r=0}^{\infty} \frac{1}{2^r} = 1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8},$$

which is 1 plus a convergent geometric series. Hence the factorial series converges (in fact, to e).

b) Consider the harmonic series

$$\sum_{r=1}^{\infty} \frac{1}{r} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

We group together the terms as follows:

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots$$

We can then see that each bracket is $\geq 1/2$, so the series is bigger than

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

which is divergent. Hence the harmonic series diverges.

c) Consider the series

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

We compare this with the series $\sum_{r=1}^{\infty} a_r$, where $a_1 = 1$ and

$$a_r = \int_{r-1}^{r} \frac{1}{x^2}\, dx$$

for $r \geq 2$. This is a convergent series, since the partial sums are given by

$$\begin{aligned} S_n &= 1 + \int_1^2 \frac{1}{x^2}\, dx + \int_2^3 \frac{1}{x^2}\, dx + \cdots + \int_{n-1}^{n} \frac{1}{x^2}\, dx \\ &= 1 + \int_1^n \frac{1}{x^2}\, dx \\ &= 1 + \left[\frac{-1}{x}\right]_1^n \\ &= 1 + (-\frac{1}{n} + 1) \\ &= 2 - \frac{1}{n} \to 2 \text{ as } n \to \infty. \end{aligned}$$

Now $a_r \geq \frac{1}{r^2}$ for all r, so $\sum_{r=1}^{\infty} \frac{1}{r^2}$ converges by the comparison test.

The ratio test

Let

$$l = \lim_{r\to\infty} \left| \frac{a_{r+1}}{a_r} \right|.$$

If $l < 1$ then $\sum_{r=0}^{\infty} a_r$ converges.
If $l > 1$ then $\sum_{r=0}^{\infty} a_r$ diverges.
If $l = 1$ then the ratio test tells you nothing.

Idea: if $l < 1$, choose r with $l < r < 1$: then the series is smaller than a geometric series with common ratio r, so must converge.

Examples

a)

$$\sum_{r=0}^{\infty} \frac{r^2}{3^r}.$$

We have $a_r = \frac{r^2}{3^r}$, so

$$\frac{a_{r+1}}{a_r} = \frac{(r+1)^2 3^r}{r^2 3^{r+1}} = \frac{1}{3}\frac{(r+1)^2}{r^2} \to \frac{1}{3}$$

as $r \to \infty$. Hence the series converges.

b)

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

We have $a_n = \frac{1}{n^2}$, so

$$\frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2} \to 1$$

as $n \to \infty$. Hence the ratio test does not tell us whether this series converges or diverges.

The alternating series test

Suppose each $a_r \geq 0$, $a_{r+1} \leq a_r$ for all r, and $a_r \to 0$ as $r \to \infty$. Then

$$\sum_{r=0}^{\infty} (-1)^r a_r$$

converges.

Examples The series

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges. The series

$$\sum_{r=0}^{\infty} (-1)^r e^{-r} = 1 - e^{-1} + e^{-2} - e^{-3} + \cdots$$

converges.

6.4 Power series (7.7)

Power series involve a variable x:

$$\sum_{r=0}^{\infty} a_r x^r.$$

Whether they converge or diverge can depend on the value of x.

Maclaurin series are examples of power series.

Let

$$l = \lim_{r\to\infty} \left| \frac{a_{r+1}x^{r+1}}{a_r x^r} \right| = \lim_{r\to\infty} |x| \left| \frac{a_{r+1}}{a_r} \right|.$$

By the ratio test, the power series converges if $l < 1$ and diverges if $l > 1$. That is, it converges if

$$|x| < \lim_{r\to\infty} \left| \frac{a_r}{a_{r+1}} \right|,$$

and diverges if

$$|x| > \lim_{r\to\infty} \left| \frac{a_r}{a_{r+1}} \right|.$$

Let

$$R = \lim_{r\to\infty} \left| \frac{a_r}{a_{r+1}} \right|,$$

the *radius of convergence* of the power series.

The power series converges if $-R < x < R$, and diverges if $x > R$ or $x < -R$. If $x = R$ or $x = -R$ the series may converge or diverge: we have to consider these cases separately.

Examples

a) Consider the power series

$$\sum_{r=0}^{\infty} \frac{x^r}{r!}.$$

We have $a_r = \frac{1}{r!}$, so $\frac{a_r}{a_{r+1}} = \frac{(r+1)!}{r!} = (r+1)$, so $R = \infty$. Hence the power series converges for all values of x.

b) Consider the power series

$$\sum_{r=1}^{\infty} \frac{x^r}{r}.$$

We have $a_r = \frac{1}{r}$, so $\frac{a_r}{a_{r+1}} = \frac{r+1}{r} \to 1$ as $r \to \infty$. Hence $R = 1$. The power series converges for $-1 < x < 1$, and diverges for $x > 1$ or $x < -1$. We have to check the cases $x = 1$, $x = -1$ separately.

If $x = 1$, the power series is

$$\sum_{r=1}^{\infty} \frac{1}{r},$$

which diverges. If $x = -1$, the power series is

$$\sum_{r=1}^{\infty} \frac{(-1)^r}{r},$$

which converges by the alternating series test.

Hence the power series converges if $-1 \le x < 1$, and diverges otherwise.

c) Consider the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n 2^n}.$$

We have $a_n = \frac{(-1)^n}{n2^n}$, so

$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{(n+1)2^{n+1}}{n2^n} = 2\frac{n+1}{n} \to 2$$

as $n \to \infty$. Hence $R = 2$, so the power series converges if $-2 < x < 2$, and diverges if $x < -2$ or $x > 2$.

When $x = 2$ we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges by the alternating series test. When $x = -2$, we have

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

which diverges.

Hence the power series converges if $-2 < x \le 2$, and diverges otherwise.