MATH191 Solutions September 2010

Section A

All questions are similar to homework problems.

1. To find the inverse function,

$$y = f(x) = \frac{2x-5}{x-1} \;\Leftrightarrow\; y(x-1) = 2x-5 \;\Leftrightarrow yx - y = 2x - 5$$

$$\Leftrightarrow x(y-2) = y-5 \;\Leftrightarrow x = \frac{y-5}{y-2}.$$

So the inverse function is given by

$$f^{-1}(y) = \frac{y+5}{y-2} \quad \text{or} \quad f^{-1}(x) = \frac{x-5}{x-2}.$$

[3 marks]

The domain of f is $(-\infty, 1) \cup (1, \infty)$, and the range of f is the domain of f^{-1}, that is, $(-\infty, 2) \cup (2, \infty)$.

[2 marks]

[$3 + 2 = 5$ marks]

2.

a) $x = 2\cos(-\pi/4) = \sqrt{2}$. (1 mark) $y = 2\sin(-\pi/4) = -\sqrt{2}$. (1 mark)

b) $r = \sqrt{9+9} = 3\sqrt{2}$ (1 mark) $\theta = \tan^{-1}(-1) = -\pi/4$ because $3 > 0$. (2 marks)

Subtract one mark for each answer not given exactly.

[$1 + 1 + 1 + 2 = 5$ marks]

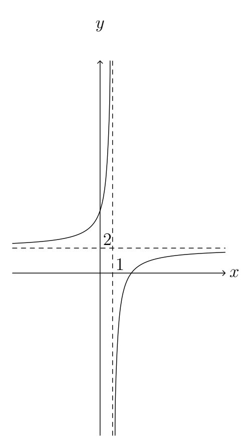
3. Since $2 \cdot 1 - 5 < 0$,

$$\lim_{x \to 1-} \frac{2x-5}{x-1} = +\infty,$$

and

$$\lim_{x \to \infty} \frac{2x-5}{x-1} = \lim_{x \to \infty} \frac{2 - \frac{5}{x}}{1 - \frac{1}{x}} = 2.$$

[3 marks]



The graph is as shown. [2 marks]

[$3 + 2 = 5$ marks]

4. In both cases, both numerator and denominator vanish at the limit, and therefore l'Hopital's Rule can be applied. In the first case we can also compute the limit simply by factorising the numerator and denominator.

a)
$$\lim_{x\to 2}\frac{x^2-x-2}{x^2-4}=\lim_{x\to 2}\frac{(x-2)(x+1)}{(x-2)(x+2)}=\lim_{x\to 2}\frac{x+1}{x+2}=\frac{3}{4}. \qquad \text{[2 marks]}$$

b) By l'Hopital's Rule
$$\lim_{x\to 0}\frac{1-\cos x}{x+\sin x}=\lim_{x\to 0}\frac{\sin x}{1+\cos x}=0. \qquad \text{[2 marks]}$$

[$2+2=4$ marks]

5.

a) By the Chain Rule,
$$\frac{d}{dx}\left(x-\ln(1+x^2)\right)=1-\frac{2x}{x^2+1}=\frac{x^2+1-2x}{1+x^2}. \qquad \text{[2 marks]}$$

b) By the quotient rule,
$$\frac{d}{dx}\left(\frac{\sin x}{x^2+1}\right)=\frac{\cos x(x^2+1)-2x\sin x}{(x^2+1)^2}. \qquad \text{[2 marks]}$$

c) By the chain rule,
$$\frac{d}{dx}\sqrt{1+3x^2}=\frac{6x}{2}(1+3x^2)^{-1/2}=\frac{3x}{\sqrt{1+3x^2}}. \qquad \text{[2 marks]}$$

[$2+2+2=6$ marks]

6. We see that
$$f'(x)=\frac{x^2+1-2x}{1+x^2}=\frac{(x-1)^2}{1+x^2}$$
So $f'(x)=0 \Leftrightarrow x=1$ and $x=1$ is the only stationary point of f. [3 marks]

Now
$$f''(x)=\frac{2(x-1)(1+x^2)-2x(x-1)^2}{(1+x^2)^2},$$
and $f''(1)=0$ so 1 is a point of inflection of f.

[2 marks: partial credit will be given if an incorrect formula for the first derivative is carried forward from question 5.]

[$3+2=5$ marks]

7.

$$\int_0^1 \left((1+x)^3 - \sqrt{1+2x} + \frac{1}{(2+x)^2}\right) \mathrm{dx}$$

$$= \left[\frac{1}{4}(1+x)^4 - \frac{2}{2\cdot 3}(1+2x)^{3/2} - (2+x)^{-1}\right]_0^1$$

$$= \left(\frac{16}{4} - \frac{3\sqrt{3}}{3} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{3} - \frac{1}{2}\right)$$

$$= \frac{11}{3} - \sqrt{3} + \frac{7}{12}$$

$$= \frac{17}{4} - \sqrt{3}.$$

[4 marks]

8. Differentiating the equation with respect to x gives

$$\frac{d}{dx}(xy^3 - 3x^2y) = y^3 + 3xy^2\frac{dy}{dx} - 6xy - 3x^2\frac{dy}{dx} = 0. \qquad \text{(2 marks)}$$

Hence

$$\frac{dy}{dx} = \frac{y^3 - 6xy}{3x^2 - 3xy^2}. \qquad \text{(2 marks)}$$

Thus $\frac{dy}{dx}$ is equal to $-\frac{11}{6}$ when $(x, y) = (2, 1)$. (2 marks)

The equation of the tangent at this point is therefore

$$y - 1 = -\frac{11}{6}(x-2) \qquad \text{or } 6y + 11x = 28. \qquad \text{(2 marks)}$$

[$2 + 2 + 2 + 2 = 8$ marks]

9.

$$\begin{aligned} z_1 + z_2 &= 1 + 3j + 2 - j = 3 + 2j, \qquad \text{(1 mark)} \\ z_1 - z_2 &= -1 + 4j, \qquad \text{(1 mark)} \\ z_1 z_2 &= (1+3j)(2-j) = 2 + 5j + 3 = 5 + 5j, \qquad \text{(2 marks)} \\ z_1/z_2 &= \frac{(1+3j)(2+j)}{2^2+1^2} = \frac{2+7j-3}{5} = \frac{-1+7j}{5} = -\frac{1}{5} + \frac{7}{5}j. \qquad \text{(2 marks)} \end{aligned}$$

[$1 + 1 + 2 + 2 = 6$ marks]

10.

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= 2\mathbf{i} - \mathbf{j} - 2\mathbf{k} + \mathbf{i} - \mathbf{j} = 3\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}, \qquad \text{(1 mark)} \\ \mathbf{a} - \mathbf{b} &= \mathbf{i} - 2\mathbf{k}, \qquad \text{(1 mark)} \\ |\mathbf{a}| &= \sqrt{2^2 + 1^2 + 2^2} = 3, \qquad \text{(1 mark)} \\ |\mathbf{b}| &= \sqrt{1^2 + 1^2} = \sqrt{2}, \qquad \text{(1 mark)} \\ \mathbf{a} \cdot \mathbf{b} &= 2 + 1 = 3. \qquad \text{(1 mark).} \end{aligned}$$

Hence the angle between $\mathbf{a}$ and $\mathbf{b}$ is θ where $\cos\theta = 3/(3\sqrt{2})$ and the angle is $\pi/4$ (2 marks).

[$1 + 1 + 1 + 1 + 1 + 2 = 7$ marks]

Section B

11.

a) The Maclaurin series expansion of e^x is

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{(3 marks)}$$

b) Hence the other Maclaurin series are

(i) for e^{x^2},

$$1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} \cdots + \frac{x^{2n}}{n!} + \cdots \qquad \text{(2 marks)}$$

(ii) for $\cosh(x)$,

$$1 + \frac{x^2}{2} + \frac{x^4}{4!} + \frac{x^6}{6!} \cdots + \frac{x^{2n}}{(2n)!} \cdots \qquad \text{(2 marks)}$$

(iii) for $\cosh \sqrt{x}$,

$$1 + \frac{x}{2} + \frac{x^2}{24} \cdots + \frac{x^n}{(2n)!} \cdots \qquad \text{(2 marks)}$$

c) Hence

$$\lim_{x\to 0} \frac{2e^{x^2} - 2\cosh(x) - x^2}{x^4}$$

$$= \lim_{x\to 0} \frac{2\left(1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} \cdots\right) - 2\left(1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} \cdots\right) - x^2}{x^4}$$

$$= \lim_{x\to 0} \frac{x^4\left(1 - \frac{1}{12}\right) + x^6\left(\frac{1}{3} - \frac{1}{360}\right) \cdots}{x^4}$$

$$= \frac{11}{12}. \qquad \text{(3 marks)}$$

d) The radius of convergence R of the Maclaurin series of $\cosh \sqrt{x}$ is given by $\lim_{n\to\infty} |a_n|/|a_{n+1}|$ (if this exists, and $+\infty$ is allowed), where $a_n = 1/(2n)!$. So

$$R = \lim_{n\to\infty} \frac{(2(n+1))!}{(2n)!} = \lim_{n\to\infty} (2n+1)(2n+2) = +\infty. \qquad \text{(3 marks)}$$

$[3 + 2 + 2 + 2 + 3 + 3 = 15 \text{ marks}]$

12a).

$$\sin\theta = \frac{1}{2} \Leftrightarrow \theta = (-1)^n \frac{\pi}{6} + n\pi,\ n \in \mathbb{Z} \qquad \text{(3 marks)}$$

b)

$$\cos(\pi - \theta) = \cos\pi\cos\theta + \sin\pi\sin\theta = -\cos\theta$$

because $\cos\pi = -1$ and $\sin\pi = 0$ (3 marks).

c)

$$2\cos\theta - 3\sin\theta = -2 \Leftrightarrow (\cos\theta + \alpha) = \frac{-2}{\sqrt{13}}$$

where $\alpha = \tan^{-1}(1.5) = 0.98279...$ Since $\alpha = \cos^{-1}(2/\sqrt{13}$, we also have $\pi - \alpha = -2/\sqrt{13}$. So $\theta = -2\alpha + (2n+1)\pi$ or $(2n+1)\pi$, for any $n \in \mathbb{Z}$, is the general solution. (6 marks)

d) There is no solution to

$$2\cos\theta - 3\sin\theta = 4$$

because this equation is equivalent to

$$\cos(\theta + \alpha) = \frac{4}{\sqrt{13}}$$

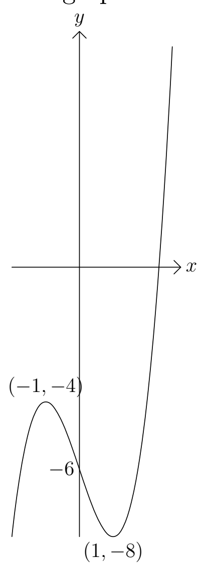
and $\sqrt{13} < 4$. (3 marks)

$[3 + 3 + 6 + 3 = 15$ marks$]$

13a). If $f(x) = x^3 - 3x - 6$, then $f'(x) = 3x^2 - 3 = 3(x-1)(x+1) = 0 \Leftrightarrow x = \pm 1$.(2 marks)

Since f' is > 0 on $(-\infty, -1) \cup (1, \infty)$ and < 0 on $(-1, 1)$, the function f is strictly increasing on each of the intervals $(-\infty, -1]$ and $[1, \infty)$, and strictly decreasing on $[-1, 1]$. But $f(-1) = -4$ and $f(1) = -8$, and $f(2) = -1$. So $f < 0$ on $(-\infty, 2]$ and there can be at most one zero on $(2, \infty)$ because f is strictly increasing on this interval. There is a zero because $f(3) = 12 > 0$. [3 marks]

The graph is as shown. [2 marks]



The Newton-Raphson formula becomes

$$x_{n+1} = x_n - \frac{x_n^3 - 3x_n - 6}{3x_n^2 - 3} = \qquad \text{(2 marks)}$$

Hence

$$x_1 = 2.444.., \quad f(x_1) = 1.2729766.. \qquad \text{(1 mark)}$$
$$x_2 = 2.359158166.., \quad f(x_2) = 0.052720477.. \qquad \text{(2 mark)}$$
$$x_3 = 2.355309079.., \quad f(x_3) = 0.0001.. \qquad \text{(3 marks)}$$

with the last answer being $f(x_3)$ to one significant figure.

A suggested method for computing the x_i and $f(x_i)$ is as follows, starting with x_0 and using the university calculator keys :

(1) 2 sto A
This stores $x_0 = 2$ in A.

(2) alpha A x^3 +3 alpha A -6 sto B
This displays $f(x_0)$ and stores it in B.

(3) 3 alpha A x^2 + 3 sto C
This displays $f'(x_0)$ and stores it in C.

(4) $A - B \div C$ sto D
This displays $x_1 = x_0 - (f(x_0)/f'(x_0)$ and stores it in D.

(5) sto A
This then stores x_1 in A, replacing x_0. The only reason for storing in D first is that if an obvious error is spotted, it is possible to return to the stored A and redo the calculation.

$[2 + 3 + 2 + 2 + 1 + 2 + 3 = 15$ marks $]$

14. For horizontal asymptotes:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{-1}{x} = 0 = \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{1}{x-1}.$$

So $y = 0$ is a horizontal asymptote. [1 mark]

For vertical asymptotes: the only possible asymptote is $x - 1 = 0$, We have

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} 2x^2 + x = 3, \quad \lim_{x\to 1+} f(x) = \lim_{x\to 1+} \frac{1}{x-1} = +\infty.$$

So $x = 1$ is a vertical asymptote. [2 marks]

For points of continuity: the only possible discontinuities are at ± 1. 1 is certainly a discontinuity, because it is a vertical asymptote . -1 is not a discontinuity because $1 = f((-1)+) = f((-1)-)$ [3 marks]

We have

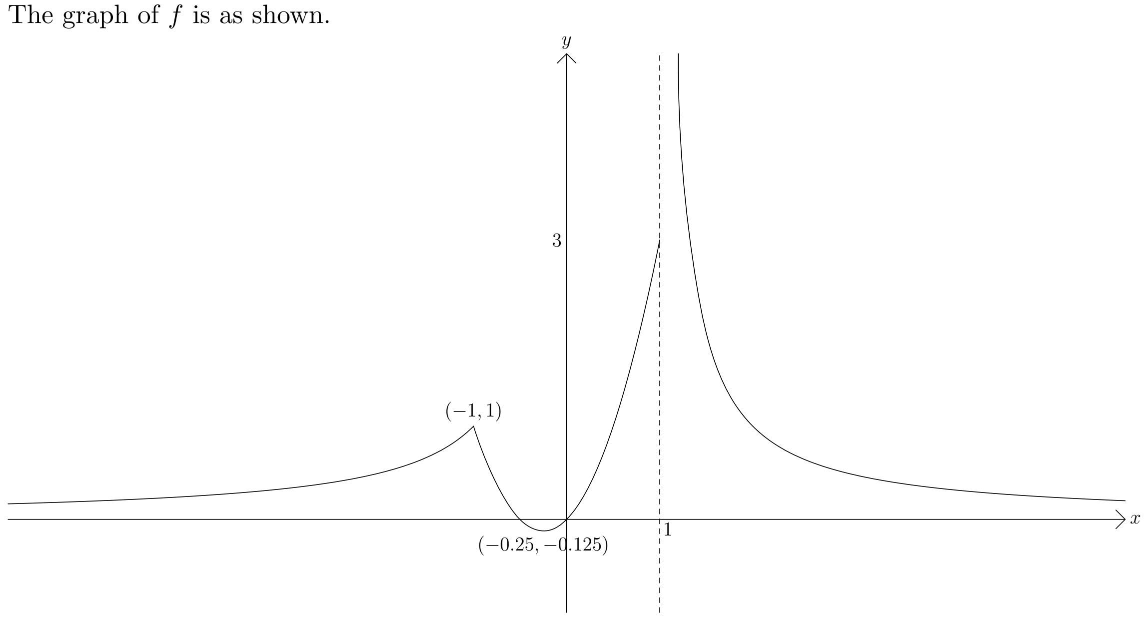
$$f'(x) = \begin{cases} x^{-2} & \text{if } x \in (-\infty, -1), \\ 4x+1 & \text{if } x \in (-1, 1), \\ -(x-1)^{-2} & \text{if } x \in (1, \infty). \end{cases}$$

[2 marks]

The function is not differentiable at 1 because it is not continuous there. It is not differentiable at -1 because $f'((-1)-) = 1 \neq f'((-1)+) = -3$. [3 marks]

By inspection we see that $f' = 0$ only at $x = -1/4$ and $f''(-1/4) = 4 > 0$. So this point is a local minimum. [2 marks]

The graph of f is as shown.



[2 marks]

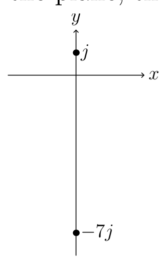
[1 + 2 + 3 + 2 + 3 + 2 + 2 = 15 marks]

15

a) By the quadratic formula, the solutions are

$$z = -3j \pm \sqrt{-9-7} = -3j \pm 4j = j \text{ or } -7j.$$

In the plane, the solutions are as shown.



[5 marks]

b) Write $z = re^{j\theta}$. The polar form of -4 is $4e^{j\pi}$. De Moivre's Theorem gives

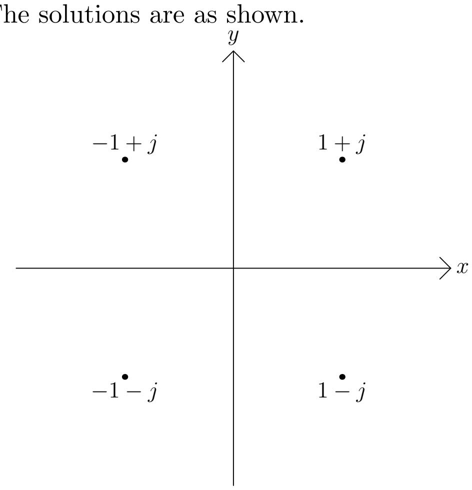
$$r^4 e^{4j\theta} = 4e^{j\pi}.$$

So $r^4 = 4$, $e^{4j\theta} = e^{j\pi}$. So $r = \sqrt{2}$ and $4\theta = \pi + 2n\pi$, any integer n. [4 marks]

Distinct values of z are given by taking $n = 0$, 1 2 and 3 that is, $\theta = \pi/4$, $\pi/4 + 2\pi/4 = 3\pi/4$, $\pi/4 + 4\pi/4 = 5\pi/4$ and $\pi/4 + 6\pi/4 = 7\pi/4$. So the solutions to $z^4 = 4$ are

$$z = \sqrt{2}\cos(\pi/4) + \sqrt{2}j\sin(\pi/4) = 1 + j,$$
$$z = \sqrt{2}\cos(3\pi/4) + \sqrt{2}j\sin(3\pi/4) = -1 + j,$$
$$z = \sqrt{2}\cos(5\pi/4) + \sqrt{2}j\sin(5\pi/4) = -1 - j,$$
$$z = \sqrt{2}\cos(7\pi/4) + \sqrt{2}j\sin(7\pi/4) = 1 - j.$$

The solutions are as shown.



[6 marks]

[5 + 4 + 2 + 2 + 2 = 15 marks]