All questions are similar to homework problems.

MATH191 Solutions September 2009
Section A

1. To find the inverse function,

$$y = \frac{x+2}{2x-1} \;\Leftrightarrow\; y(2x-1) = x+2 \;\Leftrightarrow 2yx - y = x+2$$

$$\Leftrightarrow x(2y-1) = y+2 \;\Leftrightarrow x = \frac{y+2}{2y-1}.$$

So the inverse function is given by

$$f^{-1}(y) = \frac{y+2}{2y-1} \;\text{ or }\; f^{-1}(x) = \frac{x+2}{2x-1}.$$

[3 marks]

2.

a) $r = \sqrt{3+1} = 2$ (1 mark). $\theta = -\frac{\pi}{6}$ because $\sqrt{3} > 0$ (2 marks).

b) $x = 2\cos(5\pi/3) = 1$. $y = 2\sin(5\pi/3) = -\sqrt{3}$. (1 mark each)

Subtract one mark for each answer not given exactly. [$3+2=5$ marks]

3. $\sin^{-1}(-1/\sqrt{2}) = -\pi/4$ (1 mark)
The general solution of $\sin\theta = \frac{-1}{\sqrt{2}}$ is

$$\theta = (-1)^{n+1}\pi/4 + n\pi$$

for any $n \in \mathbb{Z}$. (3 marks)
[$1+3=4$ marks]

4.

a)

$$\lim_{x\to\infty} \frac{2x^2-3x+1}{x^2+2x-1} = \lim_{x\to\infty} \frac{2-3x^{-1}+x^{-2}}{1+2x^{-1}-x^{-2}} = 2$$

(2 marks)

b)

$$\lim_{x\to(1/2)+} \frac{x+2}{2x-1} = +\infty.$$

(2 marks)

[4 marks]

5.

a) By the quotient rule,

$$\frac{d}{dx}\left(\frac{x^2+x-1}{2x-1}\right) = \frac{(2x+1)(2x-1)-2(x^2+x-1)}{(2x-1)^2} = \frac{2x^2-2x+1}{(2x-1)^2} \qquad \text{(2 marks).}$$

b) By the chain rule,

$$\frac{d}{dx}\left(e^{x^2+x}\right) = (2x+1)e^{x^2+x} \qquad \text{(2 marks).}$$

c) By the chain rule,

$$\frac{d}{dx}\ln(\cos x) = -\tan x. \qquad \text{(2 marks).}$$

[2 + 2 + 2 = 6 marks]

6.

$$\begin{aligned}\int_0^{\pi/2}(\sin(3x)+\sin^2 x)dx &= \int_0^{\pi/2}\left(\sin(3x)+\frac{1}{2}-\frac{1}{2}\cos(2x)\right)dx \\ &= \left[-\frac{1}{3}\cos(3x)+\frac{x}{2}-\frac{1}{4}\sin(2x)\right]_0^{\pi/2} \qquad \text{(3 marks)} \\ &= \frac{1}{3}+\frac{\pi}{4} \qquad \text{(2 marks)}\end{aligned}$$

[3 + 2 = 5 marks]

7. Differentiating the equation with respect to x gives

$$2y^2+4xy\frac{dy}{dx}-2xy-x^2\frac{dy}{dx}+1-2\frac{dy}{dx}=0 \qquad \text{(2 marks).}$$

Hence

$$\frac{dy}{dx}=\frac{2xy-2y^2-1}{4xy-x^2-2} \qquad \text{(2 marks).}$$

Thus $\frac{dy}{dx}$ is equal to -1 when $(x,y)=(1,1)$. (2 marks).
The equation of the tangent at this point is therefore

$$y-1=-(x-1) \qquad \text{or } y=2-x \qquad \text{(2 marks).}$$

[2 + 2 + 2 + 2 = 8 marks]

8. The domain of f is $(0,\infty)$ (1 mark).

$$f'(x)=1-\frac{2}{x}=0 \;\Leftrightarrow\; x=2 \qquad \text{(2 marks)}$$

So 2 is the only stationary point of f (1 mark)

To determine its nature,

$$f''(x) = \frac{2}{x^2}.$$

So $f''(2) = \frac{1}{2} > 0$, and 2 is a local minimum. (2 marks)

In fact 2 is a global minimum since this is the only stationary point. Hence, since $\lim_{x\to 0} f(x) = +\infty$ (or since $\lim_{x\to +\infty} f(x) = +\infty$) the range of f is $(3 - 2\ln 2, \infty)$. (2 marks: complete reasoning not required).
[1 + 2 + 1 + 2 + 2 = 8 marks]

9.

$$\begin{aligned}
z_1 + z_2 &= 2 + j \qquad \text{(1 mark)}\\
z_1 - z_2 &= 4 - 3j \qquad \text{(1 mark)}\\
z_1 z_2 &= (3 - j)(-1 + 2j) = -3 + 7j - 2j^2 = -1 + 7j \qquad \text{(2 marks)}\\
z_1/z_2 &= \frac{(3 - j)(-1 - 2j)}{(-1 + 2j)(-1 - 2j)} = \frac{-3 - 5j + 2j^2}{5} = -1 - j \qquad \text{(2 marks).}
\end{aligned}$$

[1 + 1 + 2 + 2 = 6 marks]

10.

$$\begin{aligned}
\mathbf{a} + \mathbf{b} &= 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \qquad \text{(1 mark)}\\
\mathbf{a} - \mathbf{b} &= -4\mathbf{j} + \mathbf{k} \qquad \text{(1 mark)}\\
|\mathbf{a}| &= \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14} \qquad \text{(1 mark)}\\
|\mathbf{b}| &= \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3} \qquad \text{(1 mark)}\\
\mathbf{a} \cdot \mathbf{b} &= 1 - 3 + 2 = 0 \qquad \text{(1 mark).}
\end{aligned}$$

Hence the angle between **a** and **b** is $\pi/2$ (1 mark).
[1 + 1 + 1 + 1 + 1 + 1 = 6 marks]

Section B

11.

a) The Maclaurin series expansion of $(1+x)^{-1}$ is

$$= 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots \qquad \text{(2 marks)}$$

b) The Maclaurin series expansion of $\ln(1+x)$ is

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots\cdots \qquad \text{(2 marks)}$$

Hence the other Maclaurin series are:

c) for $(1-x)^{-1}$,

$$1 - (-x) + (-x)^2 \cdots + (-1)^n(-x)^n \cdots = 1 + x + \cdots + x^n \cdots \qquad \text{(2 marks)}$$

d) for $(1+x^2)^{-1}$,

$$1 - x^2 + x^4 \cdots + (-1)^n x^{2n} \cdots \qquad \text{(2 marks)}$$

e) for $\ln(1+x^2)$,

$$x^2 - \frac{x^4}{2} \cdots + (-1)^{n+1}\frac{x^{2n}}{n} \cdots \qquad \text{(2 marks)}$$

Differentiating this term by term gives

$$2x - 2x^3 \cdots + (-1)^{n+1} 2x^{2n-1} \cdots$$

Meanwhile $2x$ times the Maclaurin series for $(1+x^2)^{-1}$ is obtained by subtracting c) from a), which gives gives

$$2x - 2x^3 \cdots + (-1)^n 2x^{2n+1} \cdot \qquad \text{(3 marks)}$$

This is to be expected because

$$\frac{d}{dx}\ln(1+x^2) = 2x(1+x^2)^{-1} \qquad \text{(2 marks)}$$

$[2+2+2+2+2+3+2 = 15$ marks$]$

12.

a) The radius of convergence R of the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

is given by

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

provided this limit exists. In this case

$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{2^{n+1} n^2}{2^n (n+1)^2} = \frac{2}{(1+1/n)^2},$$

which converges to 2. So the radius of convergence is 2. (4 marks)

At $R = 2$ the series becomes

$$\sum_{n=0}^{\infty} n^2$$

which diverges as the terms are not tending to 0. In fact, they are getting larger. At $R = -2$ the series becomes

$$\sum_{n=1}^{\infty} (-1)^n n^2.$$

which again diverges, for the same reason as above. (3 marks)

b) In this case

$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{3^n (n+1)}{3^{n+1} n} = \frac{1 + \frac{1}{n}}{3}$$

, which tends to $\frac{1}{3}$ as $n \to \infty$. Hence $R = \frac{1}{3}$. (4 marks)

At $R = \frac{1}{3}$ the series becomes

$$\sum_{n=0}^{\infty} \frac{1}{n},$$

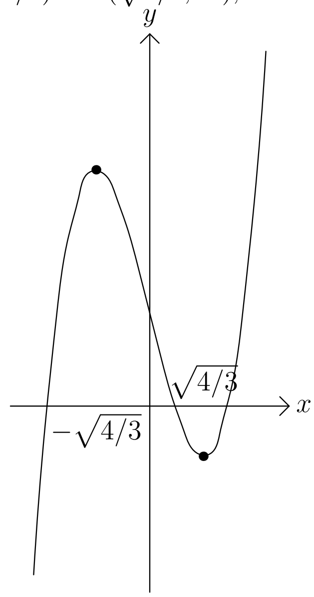
which diverges. At $R = -\frac{1}{3}$ the series becomes

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{n},$$

which is an alternating series of terms which are decreasing in modulus and tending to 0. So the series is convergent. (4 marks)

[$4 + 3 + 4 + 4 = 15$ marks]

13. For $f(x) = x^3 - 4x + 2$, $f'(x) = 3x^2 - 4 = 0 \Leftrightarrow x = \pm\sqrt{4/3}$. $f'(x) > 0$ if $x \in (-\infty, -\sqrt{4/3}) \cup (\sqrt{4/3}, \infty)$ and $f'(x) < 0$ if $x \in (-\sqrt{4/3}, \sqrt{4/3})$. So f is increasing on each of the intervals $(-\infty, -\sqrt{4/3})$ and $(\sqrt{4/3}, \infty)$, and decreasing on $(-\sqrt{4/3}, \sqrt{4/3})$. The graph is as shown.



We have

$$f(-3) = -13, \quad f(-2) = 2, \quad f(0) = 2, \quad f(1) = -1. \quad f(2) = 2.$$

Note that the local maximum $-\sqrt{4/3}$ is in the interval $(-2, 0)$ and the local minimum $\sqrt{4/3}$ is in the interval $(1, 2)$. So there must be exactly one zero in each of the intervals $(-2, -1)$, $(0, 1)$ and $(1, 2)$, and none elsewhere. (6 marks)

The Newton-Raphson formula becomes

$$x_{n+1} = x_n - \frac{x_n^3 - 4x_n + 2}{3x_n^2 - 4} = \qquad \text{(3 marks)}$$

Hence

$$x_1 = -\frac{1}{-2} = \frac{1}{2}, \quad f(x_1) = 0.125 \qquad \text{(1 mark)}$$

$$x_2 = 0.538461538, \qquad f(x_2) = 0.00227583 \qquad \text{(2 mark)}$$

$$x_3 = 0.539188599, \qquad f(x_3) = 0.000000854 \qquad \text{(2 marks)}$$

So $f(x_3)$ is 0.0000008 to 1 significant figure (1 mark)

A suggested method for computing the x_i and $f(x_i)$ is as follows, starting with x_0 and using the university calculator keys :

1. 0 sto A

   This stores $x_0 = 0$ in A.

2. alpha A x^2 +2 alpha A -2 sto B

   This displays $f(x_0)$ and stores it in B.

3. 2 alpha A + 2 sto C

   This displays $f'(x_0)$ and stores it in C.

4. $A - B \div C$ sto D

   This displays $x_1 = x_0 - (f(x_0)/f'(x_0)$ and stores it in D.

5. sto A

   This then stores x_1 in A, replacing x_0. The only reason for storing in D first is that if an obvious error is spotted, it is possible to return to the stored A and redo the calculation.

[$6 + 3 + 1 + 2 + 2 + 1 = 15$ marks]

14. For horizontal asymptotes:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (2x+1+(x-1)^{-1}) = -\infty,$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \left(1+(x-1)^{-1}\right) = 1.$$

So $y = 1$ is a horizontal asymptote (although only at $+\infty$). (2 marks)

For vertical asymptotes: the only possible asymptote is where $x - 1 = 0$, that is, where $x = 1$. We have

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} \left(3+(x-1)^{-1}\right) = -\infty, \quad \lim_{x\to 1+} \left(1+(x-1)^{-1}\right) = +\infty$$

So $x = 1$ is a vertical asymptote. (1 mark)

For points of continuity: the only possible discontinuities are at 0 and 1. 1 is certainly a discontinuity, because it is a vertical asymptote. 0 is not a discontinuity because $f(0-) = 0 = f(0+) = f(0)$. (2 marks)

We have

$$f'(x) = \begin{cases} 2-(x-1)^{-2} & \text{if } x \in (-\infty, 0), \\ -(x-1)^{-2} & \text{if } x \in (0,1) \cup (1, \infty), \end{cases}$$

The function is not differentiable at 1 because it is not continuous there. The only other point that needs checking is 0. There the function is not differentiable because the left derivative is 1 and the right derivative is -1. (3 marks)

Now $f'(x) < 0$ on each of the intervals $(0,1)$ and $(1,\infty)$. For $x \in (-\infty, 0)$, we have

$$f'(x) = 0 \iff (x-1)^2 = \frac{1}{2} \iff x = 1 \pm \sqrt{2}/2$$

Both these points are > 0 and so not in the domain of this formula for the derivative. So there are no stationary points. (2 marks)

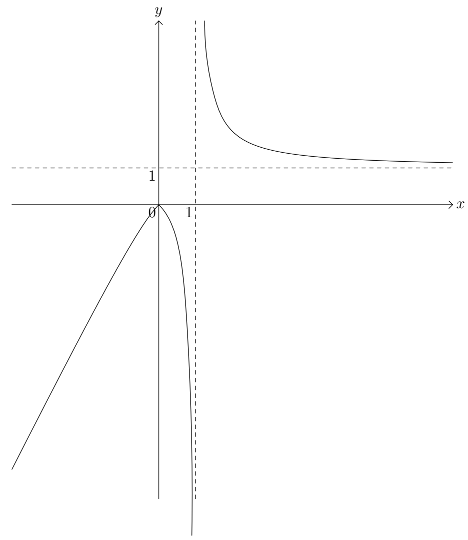
For zeros:

$$2x + 1 + \frac{1}{x-1} = 0 \iff 2x^2 - x = x(2x-1) = 0,$$

which has only the solution 0 in the set where this formula for $f(x)$ is valid and

$$1 + \frac{1}{x-1} = 0 \iff x = 0,$$

but this is the formula for f only for $x > 0$. So f has no zeros in the set where this is the valid formula for $f(x)$. So overall, 0 is the only zero of f (3 marks)



The graph of f is as shown. [2 marks]

[$2+1+2+3+3+2+2=15$ marks]

15

a) We have $1+j = \sqrt{2}e^{j\pi/4}$. So

$$(1+j)^{33} = 2^{16}\sqrt{2}e^{j(33\pi/4)} = 16394\sqrt{2}e^{j(\pi/4)} = 16394\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j\right) = 16394(1+j).$$

[5 marks]

b) Write $z = re^{j\theta}$.The polar form of $-27j$ is $27e^{3\pi j/2}$. De Moivre's Theorem gives

$$r^3 e^{3j\theta} = 27e^{3\pi j/2}.$$

So $r^3 = 27$, and $e^{3j\theta} = e^{3\pi j/2}$. So $r = 3$ and $3\theta = 3\pi/2 + 2n\pi$, any integer n.
[4 marks]

Distinct values of z are given by taking $n = 0$, 1 and 2 that is, $\theta = \pi/2$, $\pi/2 + 2\pi/3 = 7\pi/6$, and $\pi/2 + 4\pi/3 = 11\pi/2$. So the solutions to $z^3 = -27j$ are

$$z = 3j, \qquad \text{(2 marks)}$$

$$z = 3\cos(7\pi/6) + 3j\sin(7\pi/6) = -\frac{3\sqrt{3}}{2} - \frac{3}{2}j, \qquad \text{(2 marks)}$$

$$z = 3\cos(11\pi/6) + 3j\sin(11\pi/6) = \frac{3\sqrt{3}}{2} - \frac{3}{2}j. \qquad \text{(2 marks)}$$

$[4+5+2+2+2 = 15$ marks$]$