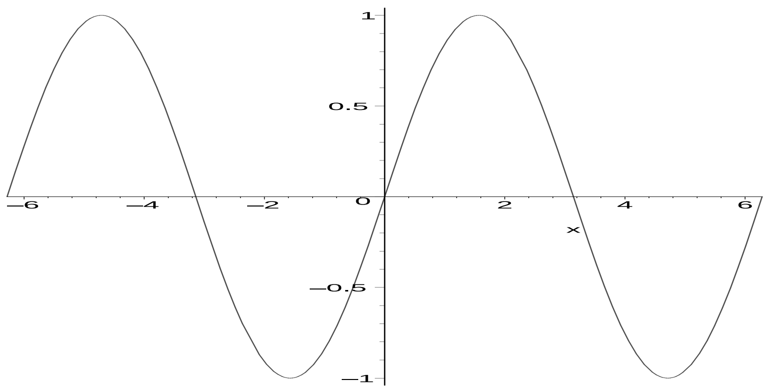
# MATH191 Exam September 2005, Solutions

All questions are standard homework examples

**1.** The maximal domain is $\mathbb{R}$ and the range is $[-1, 1]$ (1 mark each).

The graph is shown below (1 mark). It crosses the $y$-axis at $y = 0$, and the $x$-axis at $x = -2\pi, -\pi, 0, \pi$, and $2\pi$. (1 mark).




**2.** We have $f(0) = 0$, $f'(x) = 3/(1 + 3x)$, so $f'(0) = 3$, and $f''(x) = -9/(1 + 3x)^2$, so $f''(0) = -9$. (1 mark each for $f(0)$, $f'(0)$, and $f''(0)$).

Hence the Maclaurin series expansion of $f(x)$ up to the term in $x^2$ is

$$f(x) = 3x - \frac{9}{2}x^2 + \cdots$$

(1 mark for correct coefficients carried forward from $f(0)$, $f'(0)$, and $f''(0)$. 1 mark for not saying $f(x) = 3x - \frac{9}{2}x^2$).

**3.**

a) $r = \sqrt{4 + 4} = \sqrt{8}$ (1 mark). $\tan\theta = (-2)/(-2) = 1$, so since $x < 0$ we have $\theta = \tan^{-1}(1) + \pi = 5\pi/4$ (3 marks).

b) $x = \cos(5\pi/6) = -\sqrt{3}/2$. $y = \sin(5\pi/6) = 1/2$. (1 mark each)

Subtract one mark for each answer not given exactly.

**4.**

$$\begin{aligned} \int_1^4 \left(e^{-x} - \frac{1}{x}\right) dx &= \left[-e^{-x} - \ln x\right]_1^4 \\ &= (-e^{-4} - \ln(4)) - (-e^{-1} - \ln(1)) = e^{-1} - e^{-4} - \ln(4) = -1.037 \end{aligned}$$

to three decimal places. (3 marks for integration, 2 marks for substituting values and obtaining answer.)

**5.** Differentiate the defining equation with respect to $x$ to obtain

$$2x + 2y + 2x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0,$$

giving

$$\frac{dy}{dx} = -\frac{2x + 2y}{2x + 3y^2}.$$

(5 marks).

At $(x, y) = (1, 0)$ this gives $\frac{dy}{dx} = -\frac{2}{2} = -1$. (1 mark).

Hence the equation of the tangent is

$$y = 0 + (-1)(x - 1) \qquad \text{or} \qquad y = 1 - x.$$

(2 marks).

**6.**

a) By the chain rule,

$$\frac{d}{dx}\cos(x^2 + x) = -(2x + 1)\sin(x^2 + x). \qquad \text{(2 marks)}.$$

b) By the product rule,

$$\frac{d}{dx}(x^2 + x)\cos x = (2x + 1)\cos x - (x^2 + x)\sin x \qquad \text{(2 marks)}.$$

c) By the quotient rule,

$$\frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{x\cos x - \sin x}{x^2}. \qquad \text{(2 marks)}.$$

**7.** $f'(x) = 2x - \frac{2x}{x^2} = 2x - 2/x$. Stationary points are given by solutions of $f'(x) = 0$, so there are exactly two stationary points, namely $x = 1$ and $x = -1$. (3 marks, one for the equation $f'(x) = 0$, and one for each solution.)

To determine their natures, $f''(x) = 2 + 2/x^2$ so $f''(1) > 0$ and $f''(-1) > 0$. Hence both stationary points are local minima (2 marks, 1 for each stationary point.)

**8.**

$$\begin{aligned} z_1 + z_2 &= 4 + j \qquad \text{(1 mark)} \\ z_1 - z_2 &= 2 + 3j \qquad \text{(1 mark)} \\ z_1 z_2 &= (3+2j)(1-j) = 3 - 3j + 2j - 2j^2 = 5 - j \qquad \text{(2 marks)} \\ z_1/z_2 &= \frac{(3+2j)(1+j)}{(1-j)(1+j)} = \frac{1+5j}{2} \qquad \text{(2 marks).} \end{aligned}$$

**9.** $\cos^{-1}(\sqrt{3}/2) = \pi/6$ (1 mark).
Hence the general solution of $\cos\theta = \sqrt{3}/2$ is

$$\theta = \pm\frac{\pi}{6} + 2n\pi \qquad (n \in \mathbb{Z}).$$

(3 marks).

**10.**

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= 5\mathbf{i} + 2\mathbf{j} \quad \text{(1 mark)} \\ \mathbf{a} - \mathbf{b} &= -\mathbf{i} - 4\mathbf{k} \quad \text{(1 mark)} \\ |\mathbf{a}| &= \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} \qquad \text{(1 mark)} \\ |\mathbf{b}| &= \sqrt{3^2 + 1^2 + 2^2} = \sqrt{14} \qquad \text{(1 mark)} \\ \mathbf{a} \cdot \mathbf{b} &= 6 + 1 - 4 = 3 \qquad \text{(1 mark).} \end{aligned}$$

Hence the angle between **a** and **b** is $\cos^{-1}\left(\frac{3}{\sqrt{9}\sqrt{14}}\right) = 1.300$ to 3 decimal places (1 mark).

**11.** The Maclaurin series expansion of $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \text{(2 marks)}$$

Hence

a)

$$x^3 \sin x = x^4 - \frac{x^6}{3!} + \frac{x^8}{5!} - \cdots \qquad \text{(2 marks)}$$

b)

$$\sin(x^3) = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots \qquad \text{(3 marks)}$$

c)

$$\sin(2x) = 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \cdots = 2x - \frac{4x^3}{3} + \frac{4x^5}{15} - \cdots \qquad \text{(4 marks)}$$

d)

$$\begin{aligned}\sin^2 x &= \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) \\ &= x^2 - x^4\left(\frac{1}{6} + \frac{1}{6}\right) + x^6\left(\frac{1}{120} + \frac{1}{120} + \frac{1}{36}\right) \\ &= x^2 - \frac{x^4}{3} + 2\frac{x^6}{45} + \cdots \qquad \text{(4 marks).}\end{aligned}$$

**12.** The radius of the convergence $R$ of the power series

$$\sum_{n=1}^{\infty} a_n x^n$$

is given by

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|,$$

provided this limit exists.

In this case $a_n = 1/n4^n$, so $|a_n/a_{n+1}| = (n+1)4^{n+1}/(n4^n) = 4(n+1)/n$, which tends to 4 as $n \to \infty$. Hence $R = 4$. (8 marks).

When $x = -4$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

This converges by the alternating series test, which states that

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

converges if $a_n$ is a decreasing sequence with $a_n \to 0$. (3 marks)

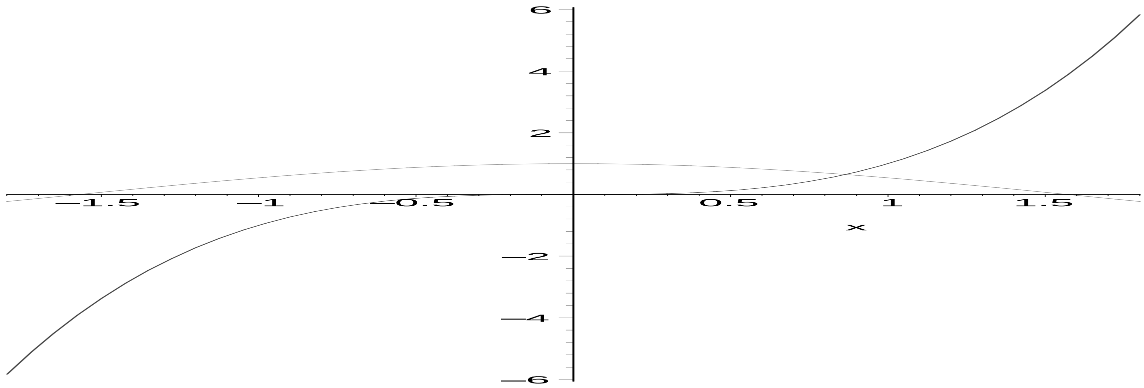
When $x = 4$, the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges (standard result) (3 marks).

Hence the series converges if and only if $-4 \le x < 4$. (1 mark).

**13.** The graphs are as shown:




(6 marks).

In $[0, \pi/2]$, $x^3$ increases strictly from 0 to $\pi^3/8 > 1$, and cos $x$ decreases strictly from 1 to 0: it follows that there is exactly one solution in $[0, \pi/2]$. In $(\pi/2, \infty)$ we have $x^3 > 1$ and cos $x \leq 1$, so there are no solutions in this interval. In $[-\pi/2, 0)$ we have $x^3 < 0$ and cos $x \geq 0$, so there are no solutions in this interval. In $(-\infty, -\pi/2)$ we have $x^3 < -1$ and cos $x \geq -1$, so there are no solutions in this interval. (3 marks).

Setting $f(x) = x^3 - \cos x$, we have $f'(x) = 3x^2 + \sin x$, so the Newton-Raphson formula becomes

$$x_{n+1} = x_n - \frac{x_n^3 - \cos x_n}{3x_n^2 + \sin x_n} \quad \text{(3 marks).}$$

Hence

$$x_1 = x_0 - \frac{x_0^3 - \cos x_0}{3x_0^2 + \sin x_0} = 0.880333.$$

$$x_2 = x_1 - \frac{x_1^3 - \cos x_1}{3x_1^2 + \sin x_1} = 0.865684.$$

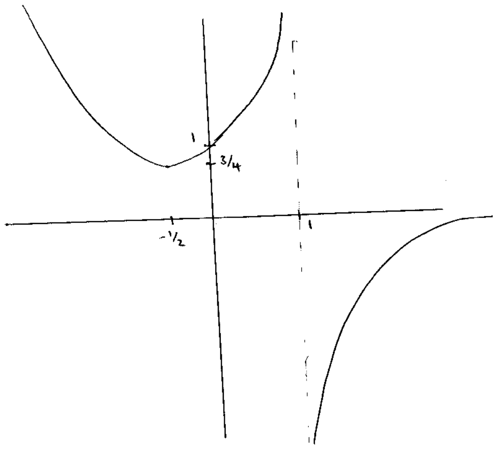
$$x_3 = x_2 - \frac{x_2^3 - \cos x_2}{3x_2^2 + \sin x_2} = 0.865474.$$

(1 mark each).

**14.** For $x \leq 0$ we have $f(x) = x^2 + x + 1$, which has no real zeros. The derivative is $f'(x) = 2x + 1$, so there is a stationary point at $x = -1/2$. Since $f''(x) = 2$, the stationary point is a local minimum. $f(x) = \frac{3}{4}$ at the stationary point. The gradient of $x^2 + x + 1$ at $x = 0$ is 1.

For $x > 0$ we have $f(x) = 1/(1 - x)$, which has no zeros and tends to 1 as $x = 0$, and to 0 as $x \to \infty$. $f'(x) = 1/(1-x)^2$, so there are no stationary points, and $f(x)$ is increasing in $(0, 1) \cup (1, \infty)$; the gradient is 1 at $x = 0$. There is a vertical asymptote at $x = 1$.

The graph of $f(x)$ is therefore




(12 marks).

$f(x)$ is not continuous at $x = 1$, since 1 is not in its maximal domain. (1 mark).

$f(x)$ is not differentiable at $x = 1$ (not in maximal domain): however, it is differentiable with derivative 1 at $x = 0$ (2 marks).

**15.** By de Moivre's theorem and the binomial theorem

$$\cos 4\theta + i \sin 4\theta = (c + is)^4 = c^4 + 4ic^3 s - 6c^2 s^2 - 4ics^3 + s^4,$$

where $c = \cos\theta$ and $s = \sin\theta$.

Equating real parts gives

$$\cos 4\theta = c^4 - 6c^2 s^2 + s^4.$$

Hence $a = 1$, $b = -6$, and $c = 1$. (5 marks)

Equating imaginary parts gives

$$\sin 4\theta = 4c^3 s - 4cs^3.$$

Hence $d = 4$ and $e = -4$. (5 marks)

When $\theta = \pi/4$, $\sin\theta = \cos\theta = 1/\sqrt{2}$, so $c^4$, $c^2s^2$, $s^4$, $c^3s$, and $cs^3$ are all equal to 1/4.

Thus the RHS of the identity for $\cos 4\theta$ becomes $1/4 - 6/4 + 1/4 = -1$, which checks since the LHS is $\cos\pi = -1$. (3 marks).

The RHS of the identity for $\sin 4\theta$ becomes $4/4 - 4/4 = 0$, which checks since the LHS is $\sin\pi = 0$. (2 marks).