Solutions to MATH191 exam January 2011

3 marks

1. To find the inverse function,

$$y = f(x) = \frac{1+2x}{1-x} \;\Leftrightarrow\; y(1-x) = 1+2x \;\Leftrightarrow y - yx = 1+2x$$
$$\Leftrightarrow y - 1 = x(2+y) \;\Leftrightarrow x = \frac{y-1}{2+y}.$$

So the inverse function is given by

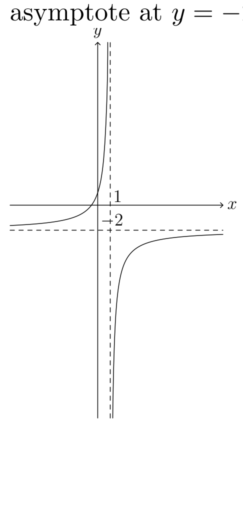
$$f^{-1}(y) = \frac{y-1}{2+y} \quad \text{or} \quad f^{-1}(x) = \frac{x-1}{2+x}.$$

2 marks

The domain of f is $(-\infty,1)\cup(1,\infty)$, and the range of f is the domain of f^{-1}, that is, $(-\infty,-2)\cup(-2,\infty)$.

2 marks

The graph of f is as shown, with vertical asymptote $x = 1$ and horizontal asymptote at $y = -2$.



Standard homework exercise
$3+2+2 = 7$ marks in total

1 mark

2a) $x = 3\cos(-3\pi/4) = -\dfrac{3}{\sqrt{2}}$.

1 mark

$y = 3\sin(-3\pi/4) = -\dfrac{3}{\sqrt{2}}$.

1 mark

b)$r = \sqrt{4+4} = 2\sqrt{2}$

2 marks

$\theta = \tan^{-1}(-1) + \pi = 3\pi/4$ because $x < 0$.

Standard homework exercise. Subtract one mark if $\sqrt{2}$ not given exactly and similarly for $3\pi/4$. $1+1+1+2 = 5$ marks

2 marks

3a) $\lim_{x\to(-1)^-}\frac{x^2+2}{x+1} = -\infty$, because $x^2+2>0$ for all x and $x+1<0$ for $x<-1$.

2 marks

b)

$$\lim_{x\to\pm\infty}\frac{2x^2-x-1}{x^2-2x+2} = \lim_{x\to\infty}\frac{2-x^{-1}-x^{-2}}{1-2x^{-1}+2x^{-2}} = 2.$$

l'Hôpital may be used
2 marks

c)

$$\lim_{x\to-1}\frac{3x^2+x-2}{4x^2+5x+1} = \lim_{x\to-1}\frac{(3x-2)(x+1)}{(4x+1)(x+1)}$$
$$= \lim_{x\to-1}\frac{3x-2}{4x+1} = \frac{5}{3}.$$

2 marks

d)Since $\tan 0 = 0^2+0 = 0$, by l'Hôpital,

$$lim_{x\to0}\frac{\tan x}{x^2+x} = \lim_{x\to0}\frac{\sec^2 x}{2x+1} = \frac{1}{1} = 1$$

Standard homework exercises
$4\times2=8$ marks

2 marks

4a) By the Product Rule, $\frac{d}{dx}(3x\sin x) = 3x\cos x + 3\sin x$.

2 marks

b) By the Quotient Rule,

$$\frac{d}{dx}\left(\frac{x+1}{x^2+3}\right) = \frac{x^2+3-2x(x+1)}{(x^2+3)^2} = \frac{3-x^2-2x}{(x^2+3)^2}$$

2 marks

c) By the Chain Rule,

$$\frac{d}{dx}\sin(x^{-1}) = -\frac{1}{x^2}\cos(x^{-1})$$

2 marks

d) By the Chain Rule

$$\frac{d}{dx}\cos(\sin x) = -\cos(x)\sin(\sin x)$$

Standard homework exercises
$4\times2=8$ marks in total

5.

3 marks

$$\int_0^{\pi/6}(\cos(3x)-\sin^2(2x))\mathrm{d}x = \int_0^{\pi/6}(\cos(3x)-\frac{1}{2}+\frac{1}{2}\cos(4x))\mathrm{d}x$$
$$= \left[\frac{1}{3}\sin(3x)-\frac{x}{2}+\frac{1}{8}\sin(4x)\right]_0^{\pi/6}$$

3 marks

$$= \frac{1}{3}\sin(\pi/2)-\frac{\pi}{12}+\frac{1}{8}\sin(2\pi/3)-0$$
$$= \frac{1}{3}-\frac{\pi}{12}+\frac{\sqrt{3}}{16}$$

Standard homework exercise
$3+3=6$ marks

6. Differentiating the equation with respect to x gives

2 marks

$$\frac{d}{dx}(4xy - y^3) = 4x\frac{dy}{dx} + 4y - 3y^2\frac{dy}{dx} = 0.$$

Hence

2 marks

$$\frac{dy}{dx} = \frac{-4y}{4x - 3y^2}.$$

Thus $\dfrac{dy}{dx}$ is equal to $\dfrac{-8}{-8} = 1$ when $(x, y) = (1, 2)$.

2 marks

2 marks

The equation of the tangent at this point is therefore

$$y - 2 = (x - 1) \qquad \text{or} \qquad y - x - 1 = 0.$$

Standard homework exercise $2 + 2 + 2 + 2 = 8$ marks

7. $z_1 + z_2 = -2 + 3j + 1 - 4j = -1 - j$ — 1 mark

$z_1 - z_2 = -2 + 3j - (1 - 4j) = -3 + 7j$ — 1 mark

$z_1 z_2 = (-2 + 3j)(1 - 4j) = -2 + 11j - 12j^2 = 10 + 11j$ — 2 marks

$z_1/z_2 = \dfrac{(-2 + 3j)(1 + 4j)}{1^2 + 4^2} = \dfrac{-2 - 5j - 12}{17} = \dfrac{-14 - 5j}{17} = -\dfrac{14}{17} - \dfrac{5}{17}j$ — 2 marks

Standard homework exercises $1 + 1 + 2 + 2 = 6$ marks

8. $\mathbf{a} + \mathbf{b} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} + 4\mathbf{i} - \mathbf{j} - \mathbf{k} = 6\mathbf{i} - 3\mathbf{k}$ — 1 mark

$\mathbf{a} - \mathbf{b} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} - (4\mathbf{i} - \mathbf{j} - \mathbf{k}) = -2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ — 1 mark

$|\mathbf{a}| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$ — 1 mark

$|\mathbf{b}| = \sqrt{4^2 + 1^2 + 1^2} = \sqrt{18} = 3\sqrt{2}$ — 1 mark

$\mathbf{a} \cdot \mathbf{b} = 8 - 1 + 2 = 9$ — 1 mark

Hence the angle between $\mathbf{a}$ and $\mathbf{b}$ is θ where — 2 marks

$$\cos\theta = \frac{9}{9\sqrt{2}} = \frac{1}{\sqrt{2}}$$

and the angle is $\pi/4$.

Standard homework exercise $1 + 1 + 1 + 1 + 1 + 2 = 7$ marks

Section B

3 marks

9a) The Maclaurin series expansion of $\dfrac{1}{1-x}$ is

$$1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$$

2 marks

b)(i) Replacing x by $-3x$, the Maclaurin series for $(1+3x)^{-1}$ is

$$1 - 3x + 9x^2 + \cdots + (-1)^n 3^n x^n + \cdots = \sum_{n=0}^{\infty} (-1)^n 3^n x^n$$

2 marks

b)(ii) Replacing x by x^3, the Maclaurin series for $(1-x^3)^{-1}$ is

$$1 + x^3 + x^6 + \cdots + x^{3n} + \cdots = \sum_{n=0}^{\infty} x^{3n}$$

5 marks

c) For $f(x) = (1-x)^{-2}$, we have $f'(x) = 2(1-x)^{-3}$, $f''(x) = (2 \times 3)(1-x)^{-4}$ and $f^{(n)}(x) = (n+1)!(1-x)^{-(n+2)}$. So

$$\frac{f^n(0)}{n!} = n+1$$

and the Maclaurin series is

$$1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots = \sum_{n=0}^{\infty} (n+1)x^n$$

Differentiating the series in a) to obtain this series will be allowed, although they have not been told about this in lectures.

3 marks

d) The radius of convergence R is $\lim_{n\to\infty} |a_n|/|a_{n+1}|$ if this exists, where a_n is the coefficient of x^n in the Maclaurin series, that is, $a_n = n+1$. So

$$R = \lim_{n\to\infty} \frac{n+1}{n+2} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n}} = 1$$

Standard homework exercises, with the last part being a much later exercise than the others.
$3 + 2 + 2 + 5 + 3 = 15$ marks

2 marks

10a) $f'(x) = 1 + \cos x = 0 \Leftrightarrow \cos x = -1 \Leftrightarrow x = (2n+1)\pi$, for $n \in \mathbb{Z}$. So the stationary points of f are all the points $(2n+1)\pi$ for $n \in \mathbb{Z}$.

2 marks

$f''(x) = -\sin x$ and $\sin((2n+1)\pi) = 0$ for all n. So these points are all points of inflection.

2 marks

Since $\cos x \geq -1$ for all x we see that $f' \geq 0$ for all x with zeros only at the points $(2n+1)\pi$ for $n \in \mathbb{Z}$. So f is strictly increasing.

3 marks

b)

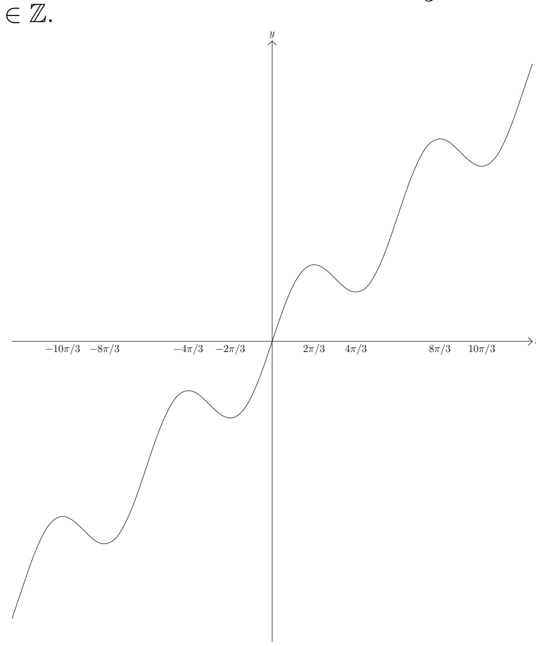
$$g'(x) = 1 + 2\cos x = 0 \quad \Leftrightarrow \quad \cos x = -\frac{1}{2} \quad \Leftrightarrow \quad x = \pm\frac{2\pi}{3} + 2n\pi, \quad n \in \mathbb{Z}$$

So the stationary points of g are $\pm\frac{2\pi}{3} + 2n\pi$ for $n \in \mathbb{Z}$

3 marks

$$g''(x) = -2\sin x \begin{cases} < 0 \text{ if } x = \frac{2\pi}{3} + 2n\pi, n \in \mathbb{Z}, \\ > 0 \text{ if } x = -\frac{2\pi}{3} + 2n\pi, n \in \mathbb{Z} \end{cases}$$

So $\frac{2\pi}{3} + 2n\pi$ is a local maximum of g for all $n \in \mathbb{Z}$ and $-\frac{2\pi}{3} + 2n\pi$ is a local minimum of g for all $n \in \mathbb{Z}$.



3 marks

The graph is as shown.

This combines computing stationary points and finding general solutions of simple trigonometric equations. Separately, these are standard homework exercises but the combination is unseen.

$2 + 2 + 2 + 3 + 3 + 3 = 15$ marks

2 marks

11a). For $f(x) = x^3 - 7x + 3$,

$$f'(x) = 3x^2 - 7 = 0 \quad \Leftrightarrow \quad x = \pm\sqrt{\frac{7}{3}}$$

3 marks

It is also clear that $f'(x) > 0$ if $x < -\sqrt{7/3}$ or $x > \sqrt{7/3}$, and $f'(x) < 0$ if $-\sqrt{7/3} < x < \sqrt{7/3}$. So f is increasing on $(-\infty, -\sqrt{73}]$ and on $[\sqrt{7/3}, \infty)$ and decreasing on $[-\sqrt{7/3}, \sqrt{7/3}]$.

2 marks

So f has at most three zeros. It has exactly three zeros because

$$f(-3) = -3 < 0, \quad f(-2) = 9 > 0 \quad f(0) = 3 > 0, \quad f(1) = -3 < 0,$$
$$f(2) = -3 < 0, \quad f(3) = 9 > 0$$

and hence there is one zero in each of the intervals $(-3, -2)$, $(0, 1)$ and $(2, 3)$.

The Newton-Raphson formula becomes

$$x_{n+1} = x_n - \frac{x_n^3 - 7x_n + 3}{3x_n^2 - 7}.$$

Hence:

2 marks — $x_1 = \frac{3}{7} = 0.42857143$ to 8 d.p.,

2 marks — $x_2 = 0.44077758$ to 8 d.p.,

4 marks — $x_3 = 0.44080771$ to 8 d.p. and $f(x_3) = 0.000000001$ to one significant figure.

A suggested method for computing the x_i and $f(x_i)$ is as follows, starting with x_0 and using the university calculator keys :

1. 1 sto A
   This stores $x_0 = 1$ in A.
2. alpha A x^3 -7 alpha A $+3$ sto B
   This displays $f(x_0)$ and stores it in B.
3. 3 alpha A x^2 $-$ 7 sto C
   This displays $f'(x_0)$ and stores it in C.
4. $A - B \div C$ sto D
   This displays $x_1 = x_0 - (f(x_0)/f'(x_0)$ and stores it in D.
5. sto A
   This then stores x_1 in A, replacing x_0. The only reason for storing in D first is that if an obvious error is spotted, it is possible to return to the stored A and redo the calculation.

Standard homework exercise

$2 + 3 + 2 + 2 + 2 + 4 = 15$ marks

12. For horizontal asymptotes:

2 marks

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{x^2+1} = \lim_{x\to-\infty} \frac{\frac{1}{x^2}}{1+\frac{1}{x^2}} = 0,$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x^2+1 = +\infty.$$

So $y = 0$ is a horizontal asymptote (although only at $-\infty$).

1 mark

There are no vertical asymptotes because $x^2+1 > 0$ for all real x and so the domain of f is $\mathbb{R}$.

3 marks

$$f'(x) = \begin{cases} \dfrac{-8x}{(x^2+1)^2} \text{ if x} < 1 \\ \\ 2x \text{ if } x > 1 \end{cases}$$

So f is differentiable everywhere except possibly at $x = 1$.

2 marks

$$f(1-) = \lim_{x\to 1-} \frac{4}{x^2+1} = 2 = \lim_{x\to 1} x^2+1$$

So f is continuous at 1.

2 marks

$$f'(1-) = \lim_{x\to 1-} \frac{-8x}{(x^2+1)^2} = -2$$

$$f'(1+) = \lim_{x\to 1+} 2x = 2.$$

Since $-2 \neq 2$, f is not differentiable at $x = 1$

3 marks

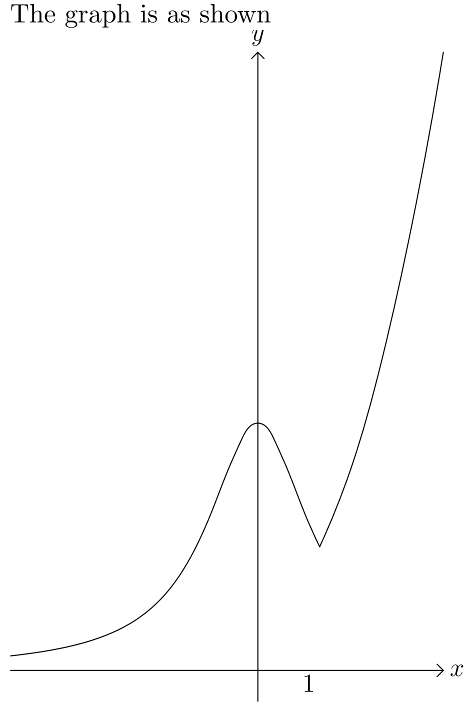
Considering the formulae for $f'(x)$ for $x < 1$ and $x > 1$, the only stationary point of f is at $x = 0$. Since, for $x < 1$,

$$f''(x) = -\frac{8}{(x^2+1)^2} + \frac{32x^2}{(x^2+1)^3}$$

we have $f''(0) = -8 < 0$ which is a local maximum.

2 marks

The graph is as shown



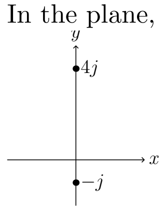
Standard homework exercise
$2+1+3+2+2+3+2 = 15$ marks

13 a) By the quadratic formula, the solutions are

5 marks

$$z = \frac{-3 \pm \sqrt{9+16}}{2j} = \frac{-3 \pm 5}{2j} = -j \text{ or } 4j.$$

In the plane, the solutions are as shown.



b) Write $z = re^{j\theta}$. The polar form of $-1+\sqrt{3}j$ is $2e^{j2\pi/3}$. De Moivre's Theorem gives

4 marks

$$r^4 e^{4j\theta} = 2e^{j2\pi/3}.$$

So $r^4 = 2$, $e^{4j\theta} = e^{j2\pi/3}$. So $r = 2^{1/4}$ and $4\theta = 2\pi/3 + 2n\pi$, any integer n. Distinct values of z are given by taking $n = 0$, 1, 2 and 3 that is, $\theta = \pi/6$, $\pi/6 + 2\pi/4 = 2\pi/3$, $\pi/6 + 4\pi/4 = 7\pi/6$ and $\pi/6 + 6\pi/4 = 5\pi/3$. So the solutions to $z^4 = 1 + \sqrt{3}j$ are

6 marks

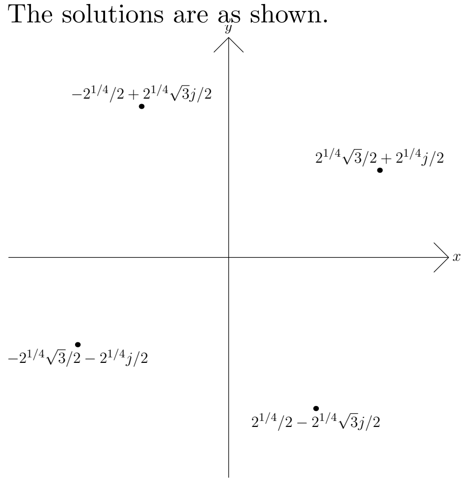
$$z = 2^{1/4}(\cos(\pi/6) + j\sin(\pi/6)) = \frac{2^{1/4}\sqrt{3}}{2} + \frac{2^{1/4}}{2}j,$$

$$z = 2^{1/4}(\cos(2\pi/3) + j\sin(2\pi/3)) = -\frac{2^{1/4}}{2} + \frac{2^{1/4}\sqrt{3}}{2}j,$$

$$z = -2^{1/4}(\cos(\pi/6) + j\sin(\pi/6)) = -\frac{2^{1/4}\sqrt{3}}{2} - \frac{2^{1/4}}{2}j$$

$$z = -2^{1/4}(\cos(2\pi/3) + j\sin(2\pi/3)) = \frac{2^{1/4}}{2} - \frac{2^{1/4}\sqrt{3}}{2}j$$

The solutions are as shown.



Standard homework exercises
$5 + 4 + 6 = 15$ marks