All questions are similar to homework problems.

MATH191 Solutions January 2009
Section A

1. To find the inverse function,

$$y = \frac{2x+1}{x+2} \Leftrightarrow y(x+2) = 2x+1 \Leftrightarrow yx + 2y = 2x+1$$

$$\Leftrightarrow x(y-2) = 1-2y \Leftrightarrow x = \frac{1-2y}{y-2}.$$

So the inverse function is given by

$$f^{-1}(y) = \frac{1-2y}{y-2} \quad \text{or} \quad f^{-1}(x) = \frac{1-2x}{x-2}.$$

[3 marks]

2.

a) $r = \sqrt{1+3} = 2$ (1 mark). $\theta = 2\frac{\pi}{3}$ because $-1 < 0$ (2 marks).

b) $x = \cos(3\pi/4) = -1/\sqrt{2}$. $y = \sin(3\pi/4) = 1/\sqrt{2}$. (1 mark each)

Subtract one mark for each answer not given exactly. [$3+2=5$ marks]

3. $\tan^{-1}(-1/\sqrt{3}) = -\pi/6$ (1 mark)
The general solution of $\tan\theta = \frac{-1}{\sqrt{3}}$ is

$$\theta = -\pi/6 + n\pi$$

for any $n \in \mathbb{Z}$. (3 marks)
[$1+3=4$ marks]

4.

a)

$$\lim_{x\to\infty} \frac{2x^2 - x + 1}{x^2+2} = \lim_{x\to\infty} \frac{2 - x^{-1} + x^{-2}}{1 + 2x^{-2}} = 2$$

(2 marks)

b)

$$\lim_{x\to(-2)+} \frac{2x+1}{x+2} = -\infty.$$

(2 marks)

[4 marks]

5.

a) By the quotient rule,

$$\frac{d}{dx}\left(\frac{x^2+1}{x-1}\right) = \frac{2x(x-1)-(x^2+1)}{(x-1)^2} = \frac{x^2-2x-1}{(x-1)^2} \qquad \text{(2 marks).}$$

b) By the product rule and chain rule,

$$\frac{d}{dx}\left(xe^{x^2}\right) = e^{x^2} + 2x^2e^{x^2} = (1+2x^2)e^{x^2} \qquad \text{(2 marks).}$$

c) By the chain rule,

$$\frac{d}{dx}\ln(x^2+3x+3) = \frac{2x+3}{x^2+3x+3}. \qquad \text{(2 marks).}$$

[$2+2+2=6$ marks]

6.

$$\begin{aligned}\int_0^{\pi/2}(\sin(2x)+\cos^2(x))dx &= \int_0^{\pi/2}\left(\sin(2x)+\frac{1}{2}+\frac{1}{2}\cos(2x)\right)dx \\ &= \left[-\frac{1}{2}\cos(2x)+\frac{x}{2}+\frac{1}{4}\sin(2x)\right]_0^{\pi/2} \qquad \text{(3 marks)} \\ &= 1+\frac{\pi}{4} \qquad \text{(2 marks)}\end{aligned}$$

[$3+2=5$ marks]

7. Differentiating the equation with respect to x gives

$$y^2 + 2xy\frac{dy}{dx} + 2xy + x^2\frac{dy}{dx} + 1 + \frac{dy}{dx} = 0 \qquad \text{(2 marks).}$$

Hence

$$\frac{dy}{dx} = -\frac{y^2+2xy+1}{2xy+x^2+1} \qquad \text{(2 marks).}$$

Thus $\frac{dy}{dx}$ is equal to -1 when $(x,y)=(1,1)$. (2 marks).
The equation of the tangent at this point is therefore

$$y-1 = -(x-1) \qquad \text{or } y = 2-x \qquad \text{(2 marks).}$$

[$2+2+2+2=8$ marks]

8. The domain of f is $(0,\infty)$ (1 mark).

$$f'(x) = 1-\frac{3}{x} = 0 \iff x = 3 \qquad \text{(2 marks)}$$

So 3 is the only stationary point of f (1 mark)

To determine its nature,

$$f''(x) = \frac{3}{x^2}.$$

So $f''(3) = \frac{1}{3} > 0$, and 3 is a local minimum. (2 marks)

In fact 3 is a global minimum since this is the only stationary point. Hence, since $\lim_{x\to 0} f(x) = +\infty$ (or since $\lim_{x\to+\infty} f(x) = +\infty$) the range of f is $(3 - 3\ln 3, \infty)$. (2 marks: complete reasoning not required).
[$1 + 2 + 1 + 2 + 2 = 8$ marks]

9.

$$\begin{aligned}
z_1 + z_2 &= 1 - j \qquad \text{(1 mark)} \\
z_1 - z_2 &= 3 - 7j \qquad \text{(1 mark)} \\
z_1 z_2 &= (2-4j)(-1+3j) = -2 + 10j - 12j^2 = 10 + 10j \qquad \text{(2 marks)} \\
z_1/z_2 &= \frac{(2-4j)(-1-3j)}{(-1+3j)(-1-3j)} = \frac{-2-2j+12j^2}{10} = \frac{-14-2j}{10} = -\frac{7}{5} - \frac{1}{5}j \qquad \text{(2 marks).}
\end{aligned}$$

[$1 + 1 + 2 + 2 = 6$ marks]

10.

$$\begin{aligned}
\mathbf{a} + \mathbf{b} &= 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k} \qquad \text{(1 mark)} \\
\mathbf{a} - \mathbf{b} &= \mathbf{i} - 5\mathbf{j} + \mathbf{k} \qquad \text{(1 mark)} \\
|\mathbf{a}| &= \sqrt{2^2 + 1^2 + 2^2} = 3 \qquad \text{(1 mark)} \\
|\mathbf{b}| &= \sqrt{1^2 + 4^2 + 1^2} = \sqrt{18} \qquad \text{(1 mark)} \\
\mathbf{a} \cdot \mathbf{b} &= 2 - 4 + 2 = 0 \qquad \text{(1 mark).}
\end{aligned}$$

Hence the angle between **a** and **b** is $\pi/2$ (1 mark).
[$1 + 1 + 1 + 1 + 1 + 1 = 6$ marks]

Section B

11.

a) The Maclaurin series expansion of $(1+x)^{-1}$ is

$$= 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots \qquad \text{(2 marks)}$$

b) The Maclaurin series expansion of $\ln(1+x)$ is

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots\cdots \qquad \text{(2 marks)}$$

Hence the other Maclaurin series are:

c) for $(1-x)^{-1}$,

$$1 - (-x) + (-x)^2 \cdots + (-1)^n(-x)^n \cdots = 1 + x + \cdots + x^n + \cdots \qquad \text{(2 marks)}$$

d) for $\ln(1-x)$,

$$-x - \frac{(-x)^2}{2} + \cdots + (-1)^{n+1}\frac{(-x)^n}{n} \cdots = -x - \frac{x^2}{2} \cdots - \frac{x^n}{n} - \cdots \qquad \text{(2 marks)}$$

e) for $\ln(1-x^2)$,

$$-x^2 - \frac{x^4}{2} \cdots - \frac{x^{2n}}{n} \cdots - \cdots \qquad \text{(2 marks)}$$

Differentiating this term by term gives

$$-2x - 2x^3 \cdots - 2x^{2n-1} \cdots$$

Meanwhile the Maclaurin series for $(1+x)^{-1} - (1-x)^{-1}$ is obtained by subtracting c) from a), which gives

$$1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots - (1 + x + \cdots + x^n \cdots)$$

$$= -2x - 2x^3 \cdots - 2x^{2n-1} \cdots \qquad \text{(3 marks)}$$

This is to be expected because

$$\frac{d}{dx}\ln(1-x^2) = \frac{d}{dx}(\ln(1+x) + \ln(1-x)) = (1+x)^{-1} - (1-x)^{-1} \qquad \text{(2 marks)}$$

[2 + 2 + 2 + 2 + 2 + 3 + 2 = 15 marks]

12.

a) The radius of convergence R of the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

is given by

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

provided this limit exists. In this case

$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{2^n(n+1)^2}{2^{n+1}n^2} = \frac{(1+1/n)^2}{2},$$

which converges to $\frac{1}{2}$. So the radius of convergence is $\frac{1}{2}$. (4 marks)

At $R = \frac{1}{2}$ the series becomes

$$\sum_{n=0}^{\infty} \frac{1}{n^2}$$

which converges. At $R = -\frac{1}{2}$ the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

which again converges, by comparison with the first series. (3 marks)

b) In this case

$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{2^n(3^{n+1}+1)}{2^{n+1}(3^n+1)} = \frac{3+3^{-n}}{2(1+3^{-n})},$$

which tends to $\frac{3}{2}$ as $n \to \infty$. Hence $R = \frac{3}{2}$. (4 marks)

At $R = \frac{3}{2}$ the series becomes

$$\sum_{n=0}^{\infty} \frac{3^n}{3^n+1} = \sum_{n=1}^{\infty} \frac{1}{1+3^{-n}}$$

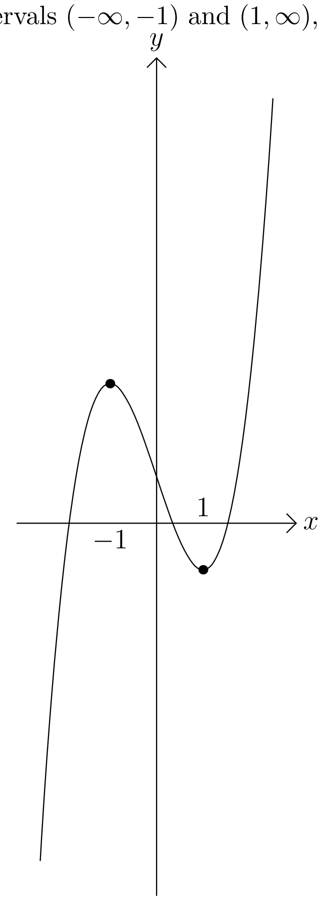
which diverges, because the terms in the sum are are tending to 1, not to 0.
At $R = -\frac{3}{2}$ the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{3^n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{1+3^{-n}}$$

which again diverges as the terms are tending to 1. (4 marks)

[$4+3+4+4 = 15$ marks]

13. For $f(x) = x^3 - 3x + 1$, $f'(x) = 3x^2 - 3 = 3(x-1)(x+1) = 0 \Leftrightarrow x = \pm 1$. Now $x - 1$ and $x + 1$ have the same sign if $x < -1$ or $x > 1$, and opposite signs if $-1 < x < 1$. So f is increasing on each of the intervals $(-\infty, -1)$ and $(1, \infty)$,



and decreasing on $(-1, 1)$. The graph is as shown.
We have

$$f(-2) = -1, \quad f(-1) = 4, \quad f(0) = 1, \quad f(1) = -1. \quad f(2) = 3.$$

So there must be exactly one zero in each of the intervals $(-2, -1)$, $(0, 1)$ and $(1, 2)$, and none elsewhere. (6 marks)

The Newton-Raphson formula becomes

$$x_{n+1} = x_n - \frac{x_n^3 - 3x_n + 1}{3x_n^2 - 3} = \qquad \text{(3 marks)}$$

Hence

$$x_1 = -\frac{1}{-3} = \frac{1}{3}, \quad f(x_1) = 0.037037037 \qquad \text{(1 mark)}$$

$$x_2 = 0.347222222, \qquad f(x_2) = 0.00019558 \qquad \text{(2 mark)}$$

$$x_3 = 0.347296353, \qquad f(x_3) = 0.0000000005 \qquad \text{(3 marks)}$$

with the last answer being $f(x_3)$ to one significant figure.

A suggested method for computing the x_i and $f(x_i)$ is as follows, starting with x_0 and using the university calculator keys :

1. 0 sto A

   This stores $x_0 = 0$ in A.

2. alpha A x^2 +2 alpha A -2 sto B

   This displays $f(x_0)$ and stores it in B.

3. 2 alpha A + 2 sto C

   This displays $f'(x_0)$ and stores it in C.

4. $A - B \div C$ sto D

   This displays $x_1 = x_0 - (f(x_0)/f'(x_0)$ and stores it in D.

5. sto A

   This then stores x_1 in A, replacing x_0. The only reason for storing in D first is that if an obvious error is spotted, it is possible to return to the stored A and redo the calculation.

[$6 + 3 + 1 + 2 + 3 = 15$ marks]

14. For horizontal asymptotes:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (x+1+2(x-1)^{-1}) = -\infty,$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \left(1+2(x-1)^{-1}\right) = 1.$$

So $y = 1$ is a horizontal asymptote (although only at $+\infty$). (2 marks)

For vertical asymptotes: the only possible asymptote is where $x - 1 = 0$, that is, where $x = 1$. We have

$$\lim_{x\to1-} f(x) = \lim_{x\to1-} \left(1+2(x-1)^{-1}\right) = -\infty, \quad \lim_{x\to1+} \left(1+2(x-1)^{-1}\right) = +\infty$$

So $x = 1$ is a vertical asymptote. (1 mark)

For points of continuity: the only possible discontinuities are at 0 and 1. 1 is certainly a discontinuity, because it is a vertical asymptote. 0 is not a discontinuity because $f(0-) = -1 = f(0+) = f(0)$. (2 marks)

We have

$$f'(x) = \begin{cases} 1-2(x-1)^{-2} & \text{if } x \in (-\infty,0), \\ -2(x-1)^{-2} & \text{if } x \in (0,1)\cup(1,\infty), \end{cases}$$

The function is not differentiable at 1 because it is not continuous there. The only other point that needs checking is 0. There the function is not differentiable because the left derivative is -1 and the right derivative is -2. (2 marks)

Now $f'(x) < 0$ on each of the intervals $(0,1)$ and $(1,\infty)$. For $x \in (-\infty,0)$, we have

$$f'(x) = 0 \;\Leftrightarrow\; (x-1)^2 = 2 \;\Leftrightarrow x = 1 \pm \sqrt{2}$$

Since $1+\sqrt{2} > 0$ and $1-\sqrt{2} < 0$, the only stationary point is $1-\sqrt{2}$. (2marks)

For $x \in (-\infty,0)$, we have

$$f''(x) = 4(x-1)^{-3}$$

and hence $f''(1-\sqrt{2}) = -\sqrt{2} < 0$, and $1-\sqrt{2}$ is a local maximum. So f is decreasing on each of the intervals $(1-\sqrt{2},1)$ and $(1,\infty)$, and increasing on $(-\infty,1-\sqrt{2})$. (2 marks)

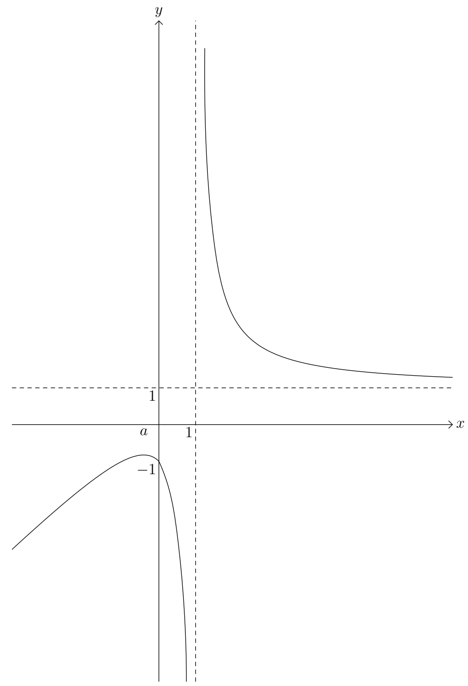
For zeros:

$$x+1+\frac{2}{x-1} = 0 \;\Leftrightarrow x^2-1+2 = 0,$$

which has no solutions, and

$$1+\frac{2}{x-1} = 0 \;\Leftrightarrow x = -1,$$

but this is the formula for f only for $x > 0$. So f has no zeros (2 marks)



The graph of f is as shown, with $a = 1-\sqrt{2}$.
[2 marks]
[$2 + 1 + 2 + 2 + 2 + 2 + 2 + 2 = 15$ marks]

15

a) We have $1-j=\sqrt{2}e^{-j\pi/4}$. So

$$(1-j)^{15}=2^7\sqrt{2}e^{-j(15\pi/4)}=128\sqrt{2}e^{j(\pi/4)}=128\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}j\right)=128(1+j).$$

[5 marks]

b) Write $z=re^{j\theta}$.The polar form of $3\sqrt{3}j$ is $3\sqrt{3}e^{j\pi/2}$. De Moivre's Theorem gives

$$r^3e^{3j\theta}=3\sqrt{3}e^{j\pi/2}.$$

So $r^3=3\sqrt{3}$, $e^{3j\theta}=e^{j\pi/2}$. So $r=\sqrt{3}$ and $3\theta=\pi/2+2n\pi$, any integer n. [4 marks]

Distinct values of z are given by taking $n=0$, 1 and 2 that is, $\theta=\pi/6$, $\pi/6+2\pi/3=5\pi/6$, and $\pi/6+4\pi/3=3\pi/2$. So the solutions to $z^3=3\sqrt{3}j$ are

$$z=\sqrt{3}\cos(\pi/6)+\sqrt{3}j\sin(\pi/6)=\frac{3}{2}+\frac{\sqrt{3}}{2}j, \qquad \text{2 marks}$$

$$z=\sqrt{3}\cos(5\pi/6)+\sqrt{3}j\sin(5\pi/6)=-\frac{3}{2}+\frac{\sqrt{3}}{2}j, \qquad \text{2 marks}$$

$$z=\sqrt{3}\cos(3\pi/2)+\sqrt{3}j\sin(3\pi/2)=-\sqrt{3}j. \qquad \text{2 marks}$$

[$4+5+2+2+2=15$ marks]