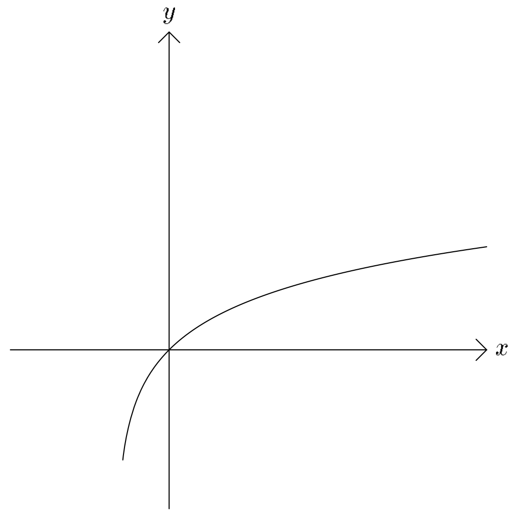
All questions are similar to homework problems.

MATH191 Solutions January 2007
Section A

1. The maximal domain is $(-1, \infty)$ and the range is $\mathbb{R}$ (1 mark each).
The graph is shown below (1 mark). It crosses the x-axis at $x = 0$ (1 mark).



[$1 + 1 + 1 + 1 = 4$ marks]

2. We have $f(0) = 2$, $f'(x) = \frac{1}{2}(4 + x)^{-1/2}$, so $f'(0) = 1/4$, and $f''(x) = -\frac{1}{4}(4 + x)^{-3/2}$, so $f''(0) = -1/32$. (1 mark each for $f(0)$, $f'(0)$, and $f''(0)$).
Hence the first three terms in the Maclaurin series expansion of $f(x)$ are

$$f(x) = 2 + x/4 - x^2/64 + \cdots.$$

(1 mark for correct coefficients carried forward from $f(0)$, $f'(0)$, and $f''(0)$. 1 mark for not saying $f(x) = 2 + x/4 - x^2/64$).
[$3 + 1 + 1 = 5$ marks]

3.

a) $r = 5$ (1 mark). $\theta = \pi$ (2 marks).

b) $x = 2\cos(3\pi/4) = 2(-1/\sqrt{2}) = -\sqrt{2}$. $y = 2\sin(\pi/4) = 2/\sqrt{2} = \sqrt{2}$. (1 mark each)

Subtract one mark for each answer not given exactly.
[$3 + 2 = 5$ marks]

4.

$$\int_1^2 e^{-x} + x^{1/2}\,dx = \left[-e^{-x} + \frac{2}{3}x^{3/2}\right]_1^2 \qquad \text{(3 marks)}$$
$$= (e^{-1} - e^{-2}) + \frac{2}{3}(2\sqrt{2} - 1). \qquad \text{(2 marks)}$$

[$3 + 2 = 5$ marks]

5. Differentiating the equation with respect to x gives

$$2x - \sin y \frac{dy}{dx} = 0 \quad \text{(2 marks)}.$$

Hence

$$\frac{dy}{dx} = \frac{2x}{\sin y} \quad \text{(2 marks)}.$$

Thus $\frac{dy}{dx}$ is equal to $2\sqrt{2}/\sqrt{3} = 2\sqrt{2/3}$ when $(x, y) = (1/\sqrt{2}, \pi/3)$. (2 marks).
The equation of the tangent at this point is therefore

$$y = 2\sqrt{2/3}x - 2/\sqrt{3} + \pi/3 \qquad \text{(2 marks)}.$$

[$2 + 2 + 2 + 2 = 8$ marks]

6.

a) By the product rule and chain rule,

$$\frac{d}{dx}(x^2 \sin 2x) = 2x\sin(2x) + 2x^2\cos(2x). \qquad \text{(2 marks)}.$$

b) By the chain rule,

$$\frac{d}{dx}(x^2 + x - 1)^{10} = 10(2x + 1)(x^2 + x - 1)^9 \qquad \text{(3 marks)}.$$

c) By the quotient rule,

$$\frac{d}{dx}\left(\frac{e^x}{x^2 + 1}\right) = \frac{e^x(x^2 + 1) - 2xe^x}{(x^2 + 1)^2}. \qquad \text{(2 marks)}.$$

[$2 + 3 + 2 = 7$ marks]

7. $f'(x) = e^x - 1$. Stationary points are given by solutions of $f'(x) = 0$, So there is exactly one stationary point, namely $x = 0$. (3 marks)

To determine its nature, $f''(x) = e^x$: so $f''(0) = 1 > 0$, and 0 is a local minimum. (2 marks)

[$3 + 2 = 5$ marks]

8.

$$\begin{aligned} z_1 + z_2 &= 5 \qquad \text{(1 mark)} \\ z_1 - z_2 &= 1 + 2j \qquad \text{(1 mark)} \\ z_1 z_2 &= (3+j)(2-j) = 6 - j - j^2 = 7 - j \qquad \text{(2 marks)} \\ z_1/z_2 &= \frac{(3+j)(2+j)}{(2-j)(2+j)} = \frac{5+5j}{5} = 1 + j \qquad \text{(2 marks).} \end{aligned}$$

[$1+1+2+2=6$ marks]

9. $\sin^{-1}(-1/\sqrt{2}) = -\pi/4$ (1 mark)
The general solution of $\sin\theta = \frac{-1}{\sqrt{2}}$ is

$$\theta = -\pi/4 + 2n\pi \text{ or } \theta = -3\pi/4 + 2n\pi$$

for any $n \in \mathbb{Z}$. (3 marks)
[$1+3=4$ marks]

10.

$$\begin{aligned} \mathbf{a}+\mathbf{b} &= 3\mathbf{i} - 4\mathbf{j} + 4\mathbf{k} \quad \text{(1 mark)} \\ \mathbf{a}-\mathbf{b} &= \mathbf{i} + 6\mathbf{j} + 2\mathbf{k} \quad \text{(1 mark)} \\ |\mathbf{a}| &= \sqrt{4^2+1^2+3^2} = \sqrt{14} \qquad \text{(1 mark)} \\ |\mathbf{b}| &= \sqrt{1^2+5^2+1^2} = \sqrt{27} \qquad \text{(1 mark)} \\ \mathbf{a}\cdot\mathbf{b} &= 2 - 5 + 3 = 0 \qquad \text{(1 mark).} \end{aligned}$$

Hence the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\pi/2$ (1 mark).
[$1+1+1+1+1+1=6$ marks]

Section B

11. The Maclaurin series expansion of e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad \text{(2 marks)}$$

Hence

a)

$$e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots \qquad \text{(2 marks)}$$

b)

$$xe^x = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots \qquad \text{(2 marks)}$$

c)

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{3!} + \cdots \qquad \text{(3 marks)}$$

d)

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \cdots \qquad \text{(3 marks)}$$

e)

$$\begin{aligned}\frac{e^x - 1}{x} - e^x &= \frac{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}{x} - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) \\ &= 1 + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \cdots - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right) \\ &= -\frac{x}{2} - \frac{x^2}{3} - \frac{x^3}{8} \cdots.\end{aligned}$$

So $f(0.1) = -0.05 - 0.003333.. - 0.000125.. = -0.053458.. = -0.0535$ to 4 decimal places. (3 marks)

$[2 + 1 + 1 + 2 + 2 + 4 + 3 = 15 \text{ marks}]$

12.

a) The radius of convergence R of the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

is given by

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

provided this limit exists. In this case $|a_n/a_{n+1}| = (n+1)^2/n^2$, which converges to 1. So the radius of convergence is 1. (4 marks)

At $R = 1$ the series becomes

$$\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which is a standard convergent series. At $R = -1$ the series becomes

$$\frac{1}{2} \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}.$$

Since the terms of the series are in modulus the same as for $R = 1$, this series is also convergent. (5 marks)

b) In this case $a_n = 1/2\sqrt{n}$, so $|a_n/a_{n+1}| = \sqrt{(n+1)/n}$, which tends to 1 as $n \to \infty$. Hence $R = 1$. (4 marks)

At $R = 1$ the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}},$$

which diverges. (2 marks)

[$4 + 5 + 4 + 2 = 15$ marks]

13. $f'(x) = 3x^2 - 3 = 0 \Leftrightarrow x = \pm 1$. Since $f'(x) = 3(x-1)(x+1)$, $f'(x) > 0$ if $x < -1$ or $x > 1$, and $f'(x) < 0$ if $x \in (-1,1)$. So f is increasing on each of the intervals $(-\infty, -1)$ and $(1, \infty)$, and decreasing on $(-1,1)$. We have

$$f(-2) = -8 + 6 - 1 = -3, \quad f(-1) = 3, \quad f(0) = 1,$$

$$f\left(\frac{1}{2}\right) = -\frac{3}{8}, \quad f(1) = -1, \quad f(2) = 3.$$

So f must change sign on each of the intervals $(-2,-1)$, $(0, \frac{1}{2})$, $(1,2)$, that is, have zeros in each of these intervals. (6 marks)

The Newton-Raphson formula becomes

$$x_{n+1} = x_n - \frac{x_n^3 - 3x_n + 1}{3x_n^2 - 3} = \frac{2x_n^3 - 1}{3x_n^2 - 3} \qquad \text{(3 marks)}$$

Hence

$$x_1 = \frac{2x_0^3 - 1}{3x_0^2 - 3} = \frac{1}{3}, \qquad \text{(1 mark)}$$

$$x_2 = \frac{2x_1^3 - 1}{3x_1^2 - 3} = \frac{25}{72} = 0.347222222, \qquad \text{(1 mark)}$$

$$x_3 = \frac{2x_2^3 - 1}{3x_2^2 - 3} = \frac{170999}{492372} = 0.347296353, \qquad \text{(2 marks)}$$

$$f(x_3) = 0.000000006. \qquad \text{(2 marks)}$$

[$6 + 3 + 1 + 1 + 2 + 2 = 15$ marks]

14. For horizontal asymptotes:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{1+x} = 0,$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \left(\frac{3}{1+x} + \frac{1}{4} - x\right) = \frac{1}{4} - \infty = -\infty.$$

So $y = 0$ is a horizontal asymptote (although only at $-\infty$). (2 marks)

For vertical asymptotes: the only possible asymptote is where $x + 1 = 0$, that is, where $x = -1$. We have

$$\lim_{x\to(-1)-} f(x) = \lim_{x\to(-1)-} \frac{1}{1+x} = -\infty$$

So $x = -1$ is a vertical asymptote, although

$$\lim_{x\to(-1)+} f(x) = \lim_{x\to(-1)+} e^x = e^{-1}. \qquad \text{(2 marks)}$$

For points of continuity: the only possible discontinuities are at -1 and 0. -1 is certainly a discontinuity, because it is a vertical asymptote. 0 is a point of continuity, because $e^0 = \frac{1}{1} = 1$. (2 marks)

For $x \in (0, \infty)$ we have

$$f(x) = \frac{\frac{3}{4}}{1+x} + \frac{1}{4} - \frac{1}{4}x.$$

So we have

$$f'(x) = \begin{cases} -\frac{1}{(1+x)^2} & \text{if } x \in (-\infty, -1) \\ e^x & \text{if } x \in (-1, 0) \\ -\frac{1}{4} - \frac{\frac{3}{4}}{(1+x)^2} & \text{if } x \in (0, \infty) \end{cases}$$

[2 marks]

The function is not differentiable at -1 because it is not continuous there. The only other point that needs checking is 0. There the function is not differentiable because the left derivative is $e^0 = 1$ and the right derivative is -1. (2 marks)

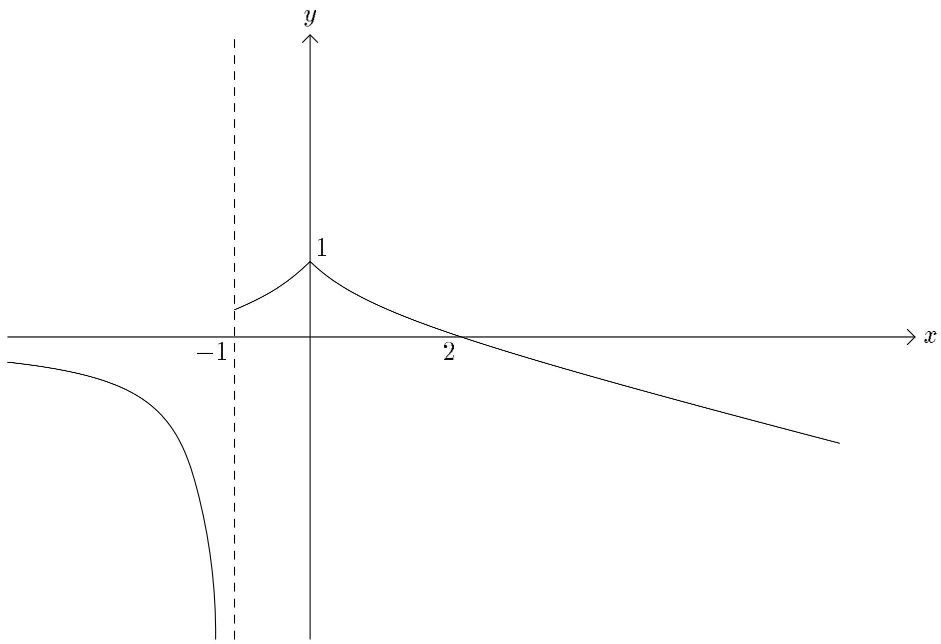
So

$$f'(x) \begin{cases} < 0 & \text{if } x \in (-\infty, -1) \\ > 0 & \text{if } x \in (-1, 0) \\ < 0 & \text{if } x \in (0, \infty) \end{cases}$$

So there are no stationary points. The function f is decreasing on each of the intervals $(-\infty, -1)$ and $[0, \infty)$ and increasing on $[-1, 0]$. (2 marks)

For zeros: since $\frac{1}{1+x} \neq 0$ for any x, and $e^x > 0$ for any x, the only possible zero is on $(0, \infty)$, when $1 - \frac{1}{4}x^2 = 0$, that is, when $x = 2$. (1 mark)

The graph of f is as shown.



[2 marks]
[$2+2+2+2+2+2+1+2=15$ marks]

15 Write $z = r\cos\theta + jr\sin\theta$.
a) The polar form of $4j$ is $4(\cos(\pi/2) + j\sin(\pi/2))$. De Moivre's Theorem gives

$$r^2(\cos 2\theta + j\sin 2\theta) = 4(\cos(\pi/2) + j\sin(\pi/2)).$$

So $r^2 = 4$, $\cos 2\theta = \cos(\pi/2)$,$\sin 2\theta = \sin(\pi/2)$. So $r = 2$ and $2\theta = \pi/2 + 2n\pi$, any integer n. So $\theta = \pi/4 + n\pi$, any integer n. So the possible values for z are

$$z = 2(\cos(\pi/4) + j\sin(\pi/4)) = \sqrt{2} + \sqrt{2}j$$

and

$$z = 2(\cos(5\pi/4) + j\sin(4\pi/4)) = -\sqrt{2} - \sqrt{2}j.$$

[6 marks]
b) The polar form of $-8j$ is $8(\cos(-\pi/2) + j\sin(-\pi/2))$. De Moivre's Theorem gives

$$r^3(\cos 3\theta + j\sin 3\theta) = 8(\cos(-\pi/2) + j\sin(-\pi/2)).$$

So $r^3 = 8$, $\cos 3\theta = \cos(-\pi/2)$, $\sin 3\theta = \sin(-\pi/2))$. So $r = 2$ and $3\theta = -\pi/2 + 2n\pi$, any integer n. Distinct values of z are given by taking $n = 0, 1, 2$, that is, $\theta = -\pi/6$, $\pi/2$, $7\pi/6$. So the solutions to $z^3 = -8j$ are

$$z = 2\cos(-\pi/6) + 2j\sin(-\pi/6) = \sqrt{3} - j,$$

$$z = 2\cos(\pi/2) + 2j\sin(\pi/2) = 2j,$$

$$z = 2\cos(7\pi/6) + 2j\sin(7\pi/6) = -\sqrt{3} - j.$$

[9 marks]
[$6 + 9 = 15$ marks]