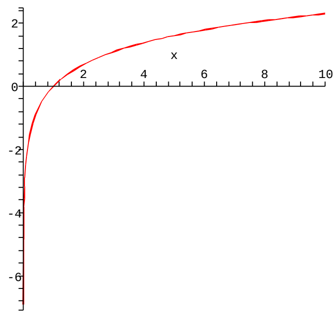
MATH191 Exam January 2005, Solutions

All questions are standard homework examples

1. The maximal domain is $(0, \infty)$ and the range is $\mathbb{R}$ (1 mark each).
The graph is shown below (1 mark). It crosses the x-axis at $x = 1$ (1 mark).



2. We have $f(0) = 2$, $f'(x) = \frac{1}{2}(4 + x)^{-1/2}$, so $f'(0) = 1/4$, and $f''(x) = -\frac{1}{4}(4 + x)^{-3/2}$, so $f''(0) = -1/32$. (1 mark each for $f(0)$, $f'(0)$, and $f''(0)$).

Hence the first three terms in the Maclaurin series expansion of $f(x)$ are

$$f(x) = 2 + x/4 - x^2/64 + \cdots.$$

(1 mark for correct coefficients carried forward from $f(0)$, $f'(0)$, and $f''(0)$. 1 mark for not saying $f(x) = 2 + x/4 - x^2/64$).

3.

a) $r = \sqrt{4+4} = \sqrt{8}$ (1 mark). $\tan\theta = 2/(-2) = -1$, so since $x < 0$ we have $\theta = \tan^{-1}(-1) + \pi = 3\pi/4$ (3 marks).

b) $x = 2\cos(7\pi/6) = 2(-\sqrt{3}/2) = -\sqrt{3}$. $y = 2\sin(7\pi/6) = 2(-1/2) = -1$. (1 mark each)

Subtract one mark for each answer not given exactly.

4.

$$\begin{aligned}\int_1^2 e^{-2x} + x^{-2}\,dx &= \left[-\frac{e^{-2x}}{2} - x^{-1}\right]_1^2 \qquad \text{(3 marks)}\\ &= (e^{-2} - e^{-4})/2 - \frac{1}{2} + 1\\ &= 0.559 \qquad \text{(2 marks)}\end{aligned}$$

to three decimal places.

5. Differentiating the equation with respect to x gives

$$3x^2 + 2xy + x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0 \quad \text{(3 marks)}.$$

Hence

$$\frac{dy}{dx} = -\frac{3x^2 + 2xy}{x^2 + 3y^2} \quad \text{(2 marks)}.$$

Thus $\frac{dy}{dx}$ is equal to -3 when $(x, y) = (1, 0)$. (1 mark).
The equation of the tangent at this point is therefore

$$y = -3x + 3 \qquad \text{(2 marks)}.$$

6.

a) By the chain rule,

$$\frac{d}{dx}\ln(1 + x^2) = \frac{2x}{1 + x^2}. \qquad \text{(2 marks)}.$$

b) By the product rule,

$$\frac{d}{dx}\left((1 + x^2)\ln x\right) = \frac{1 + x^2}{x} + 2x\ln x \qquad \text{(2 marks)}.$$

c) By the quotient rule,

$$\frac{d}{dx}\left(\frac{\cosh x}{x}\right) = \frac{x\sinh x - \cosh x}{x^2}. \qquad \text{(2 marks)}.$$

7. $f'(x) = 3x^2 - 6x = 3x(x - 2)$. Stationary points are given by solutions of $f'(x) = 0$, i.e. there are exactly two stationary points, namely $x = 0$ and $x = 2$. (3 marks, one for the equation $3x(x - 2) = 0$, and 1 for each solution.)

To determine their nature, $f''(x) = 6x - 6$: so $f''(2) > 0$, and 2 is a local minimum; while $f''(0) < 0$, and 0 is a local maximum. (2 marks, 1 for each stationary point).

8.

$$\begin{aligned}
z_1 + z_2 &= 3 - 4j \qquad \text{(1 mark)}\\
z_1 - z_2 &= -1 + 2j \qquad \text{(1 mark)}\\
z_1 z_2 &= (1 - j)(2 - 3j) = 2 - 3j - 2j + 3j^2 = -1 - 5j \qquad \text{(2 marks)}\\
z_1/z_2 &= \frac{(1 - j)(2 + 3j)}{(2 - 3j)(2 + 3j)} = \frac{5 + j}{13} \qquad \text{(2 marks)}.
\end{aligned}$$

9. One solution is $\theta_0 = \tan^{-1}(-2) = -1.107$ to three decimal places (1 mark). The general solution of $\tan\theta = -2$ is

$$\theta = \theta_0 + n\pi \qquad (n \in \mathbb{Z}) \qquad\qquad (3 \text{ marks}).$$

10.

$$\begin{aligned}
\mathbf{a}+\mathbf{b} &= 3\mathbf{i}-3\mathbf{j}-3\mathbf{k} \quad (1 \text{ mark})\\
\mathbf{a}-\mathbf{b} &= -\mathbf{i}-\mathbf{j}+5\mathbf{k} \quad (1 \text{ mark})\\
|\mathbf{a}| &= \sqrt{1^2+2^2+1^2} = \sqrt{6} \qquad (1 \text{ mark})\\
|\mathbf{b}| &= \sqrt{2^2+1^2+4^2} = \sqrt{21} \qquad (1 \text{ mark})\\
\mathbf{a}\cdot\mathbf{b} &= 2+2-4 = 0 \qquad (1 \text{ mark}).
\end{aligned}$$

Hence the angle between **a** and **b** is $\pi/2$ (1 mark).

11. The Maclaurin series expansion of $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \qquad (2 \text{ marks})$$

Hence

a)

$$x^2\cos x = x^2 - \frac{x^4}{2!} + \cdots \qquad (2 \text{ marks})$$

b)

$$\cos(x^2) = 1 - \frac{x^4}{2!} + \cdots \qquad (2 \text{ marks})$$

c)

$$\cos(2x) = 1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} - \cdots = 1 - 2x^2 + \frac{2x^4}{3} - \cdots \qquad (3 \text{ marks})$$

d)

$$\begin{aligned}
\cos^2 x &= (1+\cos(2x))/2\\
&= (2-2x^2+2x^4/3+\cdots)/2\\
&= 1 - x^2 + \frac{x^4}{3} + \cdots \qquad (4 \text{ marks}).
\end{aligned}$$

e)

$$\sin^2 x = 1 - \cos^2 x = x^2 - \frac{x^4}{3} + \cdots \qquad \text{(2 marks).}$$

12. The radius of the convergence R of the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

is given by

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

provided this limit exists. In this case $a_n = (-1)^n/(n^2+1)$, so $|a_n/a_{n+1}| = ((n+1)^2+1)/(n^2+1)$, which tends to 1 as $n \to \infty$. Hence $R = 1$. (8 marks).

When $x = 1$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}.$$

This converges by the alternating series test, which states that

$$\sum_{n=0}^{\infty} (-1)^n a_n$$

converges if a_n is a decreasing sequence with $a_n \to 0$. (3 marks)

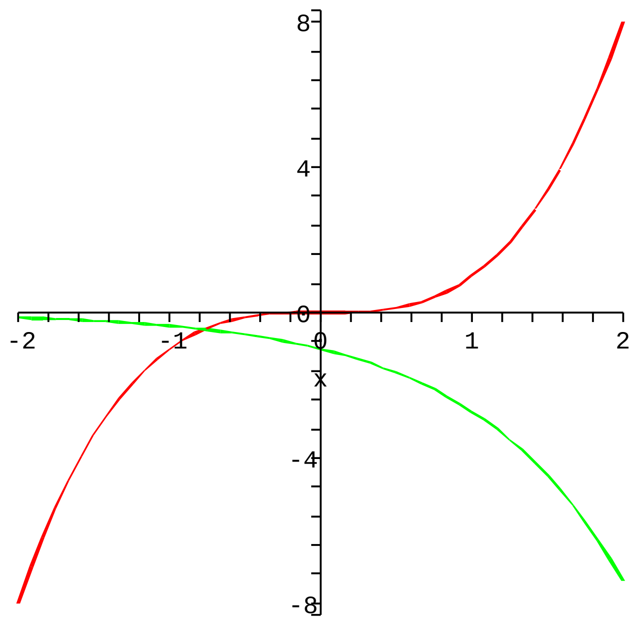
When $x = -1$, the series becomes

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1},$$

which converges (comparison with $\sum \frac{1}{n^2}$, whose convergence is a standard result). (3 marks).

Hence the series converges if and only if $-1 \le x \le 1$. (1 mark).

13. The graphs are as shown:



(6 marks).

Since x^3 increases from $-\infty$ to ∞, while $-e^x$ decreases from 0 to $-\infty$, there is exactly one value of x for which $x^3 = -e^x$. (2 marks).

Since $f(-1) = -1 + 1/e < 0$ and $f(0) = 0 + 1 > 0$, the solution to $f(x) = 0$ must lie in $[-1, 0]$. (1 mark).

We have $f'(x) = 3x^2 + e^x$, so the Newton-Raphson formula becomes

$$x_{n+1} = x_n - \frac{x_n^3 + e^{x_n}}{3x_n^2 + e^{x_n}} \quad \text{(3 marks).}$$

Hence

$$x_1 = x_0 - \frac{x_0^3 + e^{x_0}}{3x_0^2 + e^{x_0}} = -0.778097.$$

$$x_2 = x_1 - \frac{x_1^3 + e^{x_1}}{3x_1^2 + e^{x_1}} = -0.772908.$$

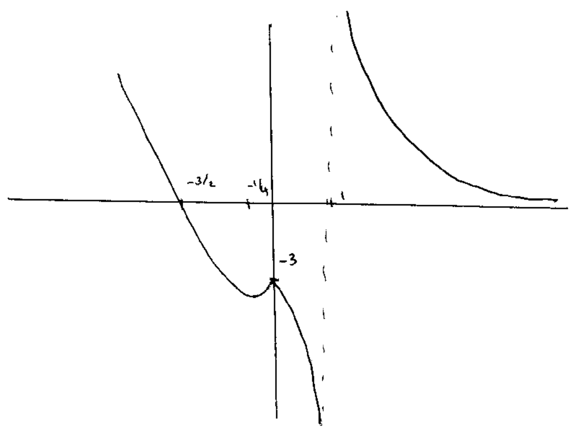
$$x_3 = x_2 - \frac{x_2^3 + e^{x_2}}{3x_2^2 + e^{x_2}} = -0.772883.$$

(1 mark each).

14. For $x < 0$ we have $f(x) = 2x^2 + x - 3 = (x-1)(2x+3)$, which has a zero at $x = -3/2$. The derivative is $f'(x) = 4x + 1$, so there is a stationary point at $x = -1/4$. Since $f''(x) = 4$, the stationary point is a local minimum. $f(x) = -3\frac{1}{8}$ at the stationary point. The gradient of $2x^2 + x - 3$ at $x = 0$ is 1.

For $x \geq 0$ we have $f(x) = 3/(x-1)$, which has no zeros and is equal to -3 at $x = 0$, and tends to 0 as $x \to \infty$. $f'(x) = -3/(x-1)^2$, so there are no stationary points, and $f(x)$ is decreasing in $(0,1) \cup (1,\infty)$; the gradient is -3 at $x = 0$. There is a vertical asymptote at $x = 1$.

The graph of $f(x)$ is therefore



(12 marks).

$f(x)$ is not continuous at $x = 1$, since 1 is not in its maximal domain. (1 mark).

$f(x)$ is not differentiable at $x = 1$ (not in maximal domain), and at $x = 0$ (no well-defined tangent to the graph at this point). (2 marks).

15. Let $z = \cos\theta + j\sin\theta$, so by de Moivre's theorem

$$\begin{aligned} z^n &= \cos n\theta + j\sin n\theta \\ z^{-n} &= \cos n\theta - j\sin n\theta. \end{aligned}$$

Thus $z^n - z^{-n} = 2j\sin n\theta$. (4 marks)

In particular, $2j\sin\theta = z - z^{-1}$ so

$$\begin{aligned} 8j^3\sin^3\theta &= (z - z^{-1})^3 \\ &= z^3 - 3z + 3z^{-1} - z^{-3} \\ &= (z^3 - z^{-3}) - 3(z - z^{-1}) \\ &= 2j\sin 3\theta - 6j\sin\theta. \end{aligned}$$

Thus, since $j^3 = -j$,

$$4\sin^3\theta = -\sin 3\theta + 3\sin\theta,$$

so $a = -1$ and $b = 3$. (5 marks)

So

$$\begin{aligned}
\int_0^\pi \sin^3 x\, dx &= \frac{1}{4}\int_0^\pi (3\sin\, x - \sin(3x))\, dx \\
&= \frac{1}{4}\left[\frac{\cos(3x)}{3} - 3\cos\, x\right]_0^\pi \\
&= \frac{1}{4}\left(\frac{\cos(3\pi) - \cos(0)}{3} - 3(\cos(\pi) - \cos(0))\right) \\
&= \frac{1}{4}\left(\frac{-2}{3} + 6\right) = 4/3.
\end{aligned}$$

(6 marks)