MATH191(R)

EXAMINER: Prof. S.M. Rees, EXTENSION 44063.

TIME ALLOWED: Three hours

Full marks are obtained by complete answers to all of Section A and THREE questions from Section B. The best three answers to Section B will be taken into account, but all answers will be marked. The marks shown against questions, or parts of questions, indicate their relative weight. Section A carries 55% of the available marks and each question in Section B is worth 15% of the available marks.

SECTION A

1. Find the inverse function of

$$f(x) = \frac{2x-5}{x-1}.$$

Determine the domain and range of f. [5 marks]

2.

a) Convert $(r, \theta) = (2, -\pi/4)$ from polar to Cartesian coordinates.

b) Convert $(x, y) = (3, -3)$ from Cartesian to polar coordinates.

In both parts, full marks will only be obtained if exact answers are given in terms of π, $\sqrt{2}$ etc..

[5 marks]

3. Let $f(x)$ be as in question 1. Compute the following. (It may be that the answer is $+\infty$ or $-\infty$.)

a) $\lim_{x\to 1-} f(x)$, b) $\lim_{x\to\infty} f(x)$.

Sketch the graph of f.

[5 marks]

4. Using l'Hopital's rule or otherwise, compute

a) $\lim_{x\to 2} \frac{x^2-x-2}{x^2-4}$, b) $\lim_{x\to 0} \frac{1-\cos x}{x+\sin x}$.

[4 marks]

5. Differentiate the following functions with respect to x:

a) $x - \ln(x^2+1)$; b) $\frac{\sin x}{x^2+1}$; c) $\sqrt{1+3x^2}$.

[6 marks]

6. For $f(x)$ as in 5a), that is, $f(x) = x - \ln(x^2+1)$, show that f has exactly one stationary point, and determine its type: that is, local minimum, local maximum or point of inflexion. [5 marks]

7. Evaluate the definite integral

$$\int_0^1 \left((1+x)^3 - \sqrt{1+2x} + \frac{1}{(2+x)^2} \right) \mathsf{dx},$$

giving your answer exactly.

[4 marks]

8. Consider the curve defined by

$$xy^3 - 3x^2y = 2.$$

Find an expression for dy/dx in terms of x and y, and hence give the equation of the tangent to the curve at the point $(x, y) = (1, 2)$.

[8 marks]

9. Let z_1 and z_2 be the complex numbers given by $z_1 = 1 + 3j$ and $z_2 = 2 - j$. Calculate $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$, and z_1/z_2.

[6 marks]

10. Let $\mathbf{a} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j}$. Find $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $|\mathbf{a}|$, $|\mathbf{b}|$, and $\mathbf{a} \cdot \mathbf{b}$. What is the angle between $\mathbf{a}$ and $\mathbf{b}$?

[7 marks]

SECTION B

11. a) Write down the Maclaurin series expansions of e^x, including the general term (you are not required to show any working if you remember the expansion).

[3 marks]

b) Hence, or otherwise, determine the Maclaurin series of

(i) e^{x^2} (ii) $\cosh x = (e^x + e^{-x})/2$ (iii) $\cosh \sqrt{x}$.

[6 marks]

c) Hence or otherwise determine

$$\lim_{x \to 0} \frac{2e^{x^2} - 2\cosh(x) - x^2}{x^4}.$$

[3 marks]

d) Find the radius of convergence of the Maclaurin series of $\cosh \sqrt{x}$.

[3 marks]

12.

a) Find all solutions of the equation

$$\sin\theta = \frac{1}{2}.$$

[3 marks]

b) Using the trigonometric formula

$$\cos(A - B) = \cos A \cos B + \sin A \sin B,$$

show that , for all θ,

$$\cos(\pi - \theta) = -\cos\theta$$

[3 marks]

c) Find all solutions of

$$2\cos\theta - 3\sin\theta = -2.$$

[6 marks]

d) Explain why the equation

$$2\cos\theta - 3\sin\theta = 4$$

has no solutions. [3 marks]

[15 marks]

13.

a) Use calculus to show that the function $f(x) = x^3 - 3x - 6$, has exactly one zero and that this zero is in $(2, 3)$. Sketch the graph. [6 marks]

b) Use the Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

with an initial guess $x_0 = 2$ to obtain successive approximations x_1, x_2, and x_3 to the solution of the equation $f(x) = 0$ in $(2, 3)$. Give x_1, x_2 and x_3 to 8 decimal places, and $f(x_3)$ to 1 significant figure.

[9 marks]

14. Let $f(x)$ be defined by

$$f(x) = \begin{cases} -\dfrac{1}{x} \text{ if } x \in (-\infty, -1), \\[2ex] 2x^2 + x \text{ if } x \in [-1, 1], \\[2ex] \dfrac{1}{x-1} \text{ if } x \in (1, \infty). \end{cases}$$

Find any horizontal and vertical asymptotes of f. Calculate $f'(x)$ where it exists. Determine whether f is continuous or not, and whether it is differentiable or not, at each of the points $x = -1$, and 1. Give the left and right derivatives at any of these points where they exist. Show that f has one stationary point and determine its type. Sketch the graph of f.

[15 marks]

15.

Find all solutions z of each of the following equations, in the form $z = a + bj$ for real a and b. In each case, indicate where the solutions are in the plane.

a) $z^2 + 6jz + 7 = 0$

[5 marks]

b) $z^4 = -4$

[10 marks]

Hint: You might find de Moivre's Theorem useful in the second part.