MATH191(R)

Examiner: Prof. S.M. Rees, Extension 44063.

Time allowed: Three hours

Answer all of Section A and THREE questions from Section B. The marks shown against questions, or parts of questions, indicate their relative weight. Section A carries 55% of the available marks.

SECTION A

1. Find the inverse function of

$$f(x) = \frac{x+2}{2x-1}.$$

[3 marks]

2.

a) Convert $(x, y) = (\sqrt{3}, -1)$ from Cartesian to polar coordinates.

b) Convert $(r, \theta) = (2, \frac{5\pi}{3})$ from polar to Cartesian coordinates.

In both parts, full marks will only be obtained if exact answers are given in terms of π, $\sqrt{3}$ etc..

[5 marks]

3. State the value of $\sin^{-1}(-1/\sqrt{2})$ (you should give an exact answer in radians, in terms of π). Give the general solution of the equation

$$\sin\theta = \frac{-1}{\sqrt{2}}.$$

[4 marks]

4. Compute the following. (It may be that the answer is $+\infty$ or $-\infty$.)

$$\text{a) } \lim_{x\to\infty} \frac{2x^2 - 3x + 1}{x^2 + 2x - 1}, \qquad \text{b) } \lim_{x\to(1/2)+} \frac{x+2}{2x-1}.$$

[4 marks]

5. Differentiate the following functions with respect to x:

$$\text{a) } \frac{x^2 + x - 1}{2x - 1}; \qquad \text{b) } e^{x^2+x}; \qquad \text{c) } \ln(\cos x).$$

[6 marks]

6. Evaluate the definite integral

$$\int_0^{\pi/2} (\sin(3x) + \sin^2 x)\mathrm{d}x,$$

giving your answer exactly in terms of π.

[5 marks]

7. Consider the curve defined by

$$2xy^2 - x^2y + x - 2y = 0.$$

Find an expression for dy/dx in terms of x and y, and hence give the equation of the tangent to the curve at the point $(x, y) = (1, 1)$.

[8 marks]

8. State the maximal domain of the function

$$f(x) = x + 1 - 2\ln x.$$

Show that f has one stationary point. Determine whether the stationary point is a local maximum, a local minimum, or a point of inflection. State the range of f.

[8 marks]

9. Let z_1 and z_2 be the complex numbers given by $z_1 = 3-j$ and $z_2 = -1+2j$. Calculate $z_1 + z_2$, $z_1 - z_2$, z_1z_2, and z_1/z_2.

[6 marks]

10. Let $\mathbf{a} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Find $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $|\mathbf{a}|$, $|\mathbf{b}|$, and $\mathbf{a} \cdot \mathbf{b}$. What is the angle between $\mathbf{a}$ and $\mathbf{b}$?

[6 marks]

SECTION B

11. Write down the Maclaurin series expansions of the following functions, including the general term (you are not required to show any working if you remember the expansions):

$$\text{a) } (1+x)^{-1}; \qquad \text{b) } \ln(1+x).$$

[4 marks]

Hence, or otherwise, determine the Maclaurin series expansions of the following functions:

$$\text{c) } (1-x)^{-1}; \qquad \text{d) } (1+x^2)^{-1}; \qquad \text{e) } \ln(1+x^2).$$

[6 marks]

Verify that differentiating the Maclaurin series of $\ln(1+x^2)$ term by term gives the Maclaurin series of $(1+x^2)^{-1}$ multiplied by $2x$, and state why you expect this to be true.

[5 marks]

12. In each of the following cases, calculate the radius of convergence R of each of the power series, and determine whether the series converges at $x = \pm R$.

$$\text{a) } \sum_{n=0}^{\infty} \frac{n^2}{2^n} x^n; \qquad \text{b) } \sum_{n=1}^{\infty} \frac{3^n}{n} x^n.$$

[15 marks]

13. Use calculus to sketch the graph of the function $f(x) = x^3 - 4x + 2$, and show that f has exactly three zeros, one in $(-3, -2)$, one in $(0, 1)$ and one in $(1, 2)$

[6 marks]

Use the Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

with an initial guess $x_0 = 0$ to obtain successive approximations x_1, x_2, and x_3 to the solution of the equation $f(x) = 0$ in $(0, 1)$. Give x_1, x_2 and x_3 to 8 decimal places, and $f(x_3)$ to 1 significant figure.

[9 marks]

14. Let $f(x)$ be defined by

$$f(x) = \begin{cases} 2x + 1 + (x-1)^{-1} & \text{if } x \in (-\infty, 0], \\ 1 + (x-1)^{-1} & \text{if } x \in (0,1) \cup (1, \infty). \end{cases}$$

Find any horizontal and vertical asymptotes of f. Identify any points of discontinuity of f. Calculate $f'(x)$ where it exists. Identify any points where f is not differentiable. Show that f has no stationary points, and determine maximal intervals on which f is decreasing and maximal intervals on which f is increasing. Find any zeros of f. Sketch the graph of f.

[15 marks]

15.

a) Compute $(1 + j)^{33}$.

b) Find all z in the form $a + bj$, for real a and b, with $z^3 = -27j$.

Hint: You might find de Moivre's Theorem useful in both parts.

[15 marks]