MATH191(R)

Examiner: Prof. S.M. Rees, Extension 44063.

Time allowed: Three hours

Answer all of Section A and THREE questions from Section B. The marks shown against questions, or parts of questions, indicate their relative weight. Section A carries 55% of the available marks.

SECTION A

1. Find the inverse function of

$$f(x) = \frac{2x+1}{x-3}.$$

Sketch the graph of $y = f(x)$, indicating the coordinates of any points where the graph crosses the axes.

[4 marks]

2. By evaluating $f(0)$, $f'(0)$, and $f''(0)$, obtain the Maclaurin series expansion of the function

$$f(x) = \ln(1 - 2x)$$

up to and including the term in x^2.

[5 marks]

3.

a) Convert $(x, y) = (-1, \sqrt{3})$ from Cartesian to polar coordinates.

b) Convert $(r, \theta) = (3, 5\pi/3)$ from polar to Cartesian coordinates.

In both parts, full marks will only be obtained if exact answers are given in terms of π, $\sqrt{3}$ etc..

[5 marks]

4. Evaluate the definite integral

$$\int_0^1 \left(\frac{1}{\sqrt{1+x}} + (1+x)^{-2} \right) \mathrm{d}x,$$

giving your answer exactly in terms of square roots.

[5 marks]

5. Consider the curve defined by

$$x^3 + xy^2 + y + 1 = 0.$$

Find an expression for dy/dx in terms of x and y, and hence give the equation of the tangent to the curve at the point $(x, y) = (-1, 1)$.

[8 marks]

6. Differentiate the following functions with respect to x:

$$\text{a) } \cos(x^3); \qquad \text{b) } \frac{\ln x}{x^2 - 2}; \qquad \text{c) } (x^2 + 2x + 2)^{1/2}.$$

[6 marks]

7. Show that the function

$$f(x) = xe^{-x}$$

has exactly one stationary point. Determine whether the stationary point is a local maximum, a local minimum, or a point of inflection.

[6 marks]

8. Let z_1 and z_2 be the complex numbers given by $z_1 = 3 + j$ and $z_2 = 1 - 4j$. Calculate $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$, and z_1/z_2.

[6 marks]

9. State the value of $\cos^{-1}(-\sqrt{3}/2)$ (you should give an exact answer in radians, in terms of π). Give the general solution of the equation

$$\cos\theta = \frac{-\sqrt{3}}{2}.$$

[4 marks]

10. Let $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$. Find $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $|\mathbf{a}|$, $|\mathbf{b}|$, and $\mathbf{a} \cdot \mathbf{b}$. What is the angle between $\mathbf{a}$ and $\mathbf{b}$?

[6 marks]

SECTION B

11. Write down the Maclaurin series expansion of the function $f(x) = \sqrt{1+x}$, giving the expression for the general term. (You are not required to show any working if you remember this expansion.)

[2 marks]

Hence, or otherwise, determine the Maclaurin series expansions of the following functions:

$$\text{a) } \sqrt{1-x} \qquad \text{b) } \sqrt{4+x}; \qquad \text{c) } \sqrt{1-x^2}$$

[8 marks]

Hence, or otherwise, determine the n'th derivative at 0, $g^{(n)}(0)$, where $g(x) = \sqrt{1-x^2}$.

[5 marks]

12. In each of the following cases, calculate the radius of convergence R of each of the power series, and determine whether the series converges at $x = \pm R$.

$$\text{a) } \sum_{n=0}^{\infty} \frac{2^n}{(n+1)^2} x^n; \qquad \text{b) } \sum_{n=0}^{\infty} \frac{1}{2^n \sqrt{n+1}} x^n.$$

[15 marks]

13. Use calculus to show that the function $f(x) = x^3 + x^2 + 2x - 3$ has exactly one zero, in (0,1).

[6 marks]

Use the Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

with an initial guess $x_0 = 1$ to obtain successive approximations x_1, x_2, and x_3 to the solution of the equation $f(x) = 0$ in $(0, 1)$. Give x_2, x_3 to 8 decimal places, and $f(x_3)$ to 1 significant figure.

[9 marks]

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14. Let $f(x)$ be defined by

$$f(x) = \begin{cases} x + (1-x)^{-1} & \text{if } x \in (-\infty, 0], \\ 1 - x & \text{if } x \in (0, 1], \\ (1-x)^{-1} & \text{if } x \in (1, \infty). \end{cases}$$

Find any horizontal and vertical asymptotes of f. Identify any points of discontinuity of f. Calculate $f'(x)$ where it exists. Identify any points where f is not differentiable. Show that f has no stationary points, and determine maximal intervals on which f is decreasing and maximal intervals on which f is increasing. Find any zeros of f. Sketch the graph of f.

[15 marks]

15.

a) Compute $(1 - j)^{15}$.

b) Find all z in the form $a + bj$, for real a and b, with $z^3 = -27$.

Hint: You might find de Moivre's Theorem useful in both parts. [15 marks]