UNIVERSITY OF LIVERPOOL

MATH191

Examiner: Prof. S.M. Rees, Extension 44063.

Time allowed: Three hours

Answer all of Section A and THREE questions from Section B. The marks shown against questions, or parts of questions, indicate their relative weight. Section A carries 55% of the available marks.

SECTION A

1. Find the inverse function of

$$f(x) = \frac{2x+1}{x+2}.$$

[3 marks]

2.

a) Convert $(x, y) = (-1, \sqrt{3})$ from Cartesian to polar coordinates.

b) Convert $(r, \theta) = (1, \frac{3\pi}{4})$ from polar to Cartesian coordinates.

In both parts, full marks will only be obtained if exact answers are given in terms of π, $\sqrt{2}$ etc..

[5 marks]

3. State the value of $\tan^{-1}(-1/\sqrt{3})$ (you should give an exact answer in radians, in terms of π). Give the general solution of the equation

$$\tan\theta = \frac{-1}{\sqrt{3}}.$$

[4 marks]

4. Compute the following. (It may be that the answer is $+\infty$ or $-\infty$.)

a) $\displaystyle\lim_{x\to\infty} \frac{2x^2 - x + 1}{x^2+2}$, b) $\displaystyle\lim_{x\to(-2)+} \frac{2x+1}{x+2}$.

[4 marks]

5. Differentiate the following functions with respect to x:

a) $\dfrac{x^2+1}{x-1}$; b) xe^{x^2}; c) $\ln(x^2+3x+3)$.

[6 marks]

6. Evaluate the definite integral

$$\int_0^{\pi/2} (\sin(2x) + \cos^2 x)\mathrm{d}x,$$

giving your answer exactly in terms of π.

[5 marks]

7. Consider the curve defined by

$$xy^2 + x^2y + x + y = 4.$$

Find an expression for dy/dx in terms of x and y, and hence give the equation of the tangent to the curve at the point $(x, y) = (1, 1)$.

[8 marks]

8. State the maximal domain of the function

$$f(x) = x - 3\ln x.$$

Show that f has one stationary point. Determine whether the stationary point is a local maximum, a local minimum, or a point of inflection. State the range of f.

[8 marks]

9. Let z_1 and z_2 be the complex numbers given by $z_1 = 2 - 4j$ and $z_2 = -1 + 3j$. Calculate $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$, and z_1/z_2.

[6 marks]

10. Let $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 4\mathbf{j} + \mathbf{k}$. Find $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $|\mathbf{a}|$, $|\mathbf{b}|$, and $\mathbf{a} \cdot \mathbf{b}$. What is the angle between $\mathbf{a}$ and $\mathbf{b}$?

[6 marks]

SECTION B

11. Write down the Maclaurin series expansions of the following functions, including the general term (you are not required to show any working if you remember the expansions):

a) $(1+x)^{-1}$;   b) $\ln(1+x)$.

[4 marks]

Hence, or otherwise, determine the Maclaurin series expansions of the following functions:

c) $(1-x)^{-1}$;   d) $\ln(1-x)$;   e) $\ln(1-x^2)$.

[6 marks]

Verify that differentiating the Maclaurin series of $\ln(1-x^2)$ term by term gives the Maclaurin series of $(1+x)^{-1}-(1-x)^{-1}$, and state why you expect this to be true.

[5 marks]

12. In each of the following cases, calculate the radius of convergence R of each of the power series, and determine whether the series converges at $x=\pm R$.

a) $\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{n^2}x^n$;   b) $\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{3^n+1}x^n$.

[15 marks]

13. Use calculus to sketch the graph of the function $f(x)=x^3-3x+1$, and show that f has three zeros, one in $(-2,-1)$, one in $(0,1)$ and one in $(1,2)$

[6 marks]

Use the Newton-Raphson formula

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

with an initial guess $x_0=0$ to obtain successive approximations x_1, x_2, and x_3 to the solution of the equation $f(x)=0$ in $(0,1)$. Give x_1, x_2 and x_3 to 8 decimal places, and $f(x_3)$ to 1 significant figure.

[9 marks]

14. Let $f(x)$ be defined by

$$f(x) = \begin{cases} x + 1 + 2(x-1)^{-1} & \text{if } x \in (-\infty, 0], \\ 1 + 2(x-1)^{-1} & \text{if } x \in (0,1) \cup (1, \infty). \end{cases}$$

Find any horizontal and vertical asymptotes of f. Identify any points of discontinuity of f. Calculate $f'(x)$ where it exists. Identify any points where f is not differentiable. Show that f has one stationary point, determine its type, and determine maximal intervals on which f is decreasing and maximal intervals on which f is increasing. Find any zeros of f. Sketch the graph of f.

[15 marks]

15.

a) Compute $(1-j)^{15}$.

b) Find all z in the form $a + bj$, for real a and b, with $z^3 = 3\sqrt{3}j$.

Hint: You might find de Moivre's Theorem useful in both parts.

[15 marks]