MATH191

EXAMINER: Prof. S.M. Rees, EXTENSION 44063.

TIME ALLOWED: Three hours

Answer all of Section A and THREE questions from Section B. The marks shown against questions, or parts of questions, indicate their relative weight. Section A carries 55% of the available marks.

SECTION A

1. Find the inverse function of

$$f(x) = \frac{x+3}{x-2}.$$

Sketch the graph of $y = f(x)$, indicating the coordinates of any points where the graph crosses the axes.

[4 marks]

2. By evaluating $f(0)$, $f'(0)$, and $f''(0)$, obtain the Maclaurin series expansion of the function

$$f(x) = (1-2x)^{-1}$$

up to and including the term in x^2.

[5 marks]

3.

a) Convert $(x, y) = (-4, -4)$ from Cartesian to polar coordinates.

b) Convert $(r, \theta) = (4\sqrt{2}, 5\pi/4)$ from polar to Cartesian coordinates.

In both parts, full marks will only be obtained if exact answers are given in terms of π, $\sqrt{2}$ etc..

[5 marks]

4. Evaluate the definite integral

$$\int_0^1 \left(\frac{1}{1+x} + (1+x)^{-2} \right) \mathrm{d}x,$$

giving your answer exactly in terms of natural logs.

[5 marks]

5. Consider the curve defined by

$$x^3 + xy + y^3 + 1 = 0.$$

Find an expression for dy/dx in terms of x and y, and hence give the equation of the tangent to the curve at the point $(x, y) = (1, -1)$.

[8 marks]

6. Differentiate the following functions with respect to x:

a) $\sin(x^3 - 1)$; b) $\dfrac{\sin x}{x^2 + 1}$; c) $\ln(x^3 + 2x - 1)$.

[6 marks]

7. Show that the function

$$f(x) = \frac{1}{1 + x^2}$$

has exactly one stationary point. Determine whether the stationary point is a local maximum, a local minimum, or a point of inflection.

[6 marks]

8. Let z_1 and z_2 be the complex numbers given by $z_1 = 1+2j$ and $z_2 = 1-3j$. Calculate $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$, and z_1/z_2.

[6 marks]

9. State the value of $\sin^{-1}(-\sqrt{3}/2)$ (you should give an exact answer in radians, in terms of π). Give the general solution of the equation

$$\sin\theta = \frac{-\sqrt{3}}{2}.$$

[4 marks]

10. Let $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 5\mathbf{j} + \mathbf{k}$. Find $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $|\mathbf{a}|$, $|\mathbf{b}|$, and $\mathbf{a} \cdot \mathbf{b}$. What is the angle between $\mathbf{a}$ and $\mathbf{b}$?

[6 marks]

SECTION B

11. Write down the Maclaurin series expansion of the function $f(x) = (1+x)^{-1}$, including the general term. (You are not required to show any working if you remember this expansion.)

[2 marks]

Hence, or otherwise, determine the Maclaurin series expansions of the following functions:

a) $(1+2x)^{-1}$; b) $(2+x)^{-1}$; c) $(1+x^2)^{-1}$.

[8 marks]

Hence, or otherwise, determine the n'th derivative at 0, $g^{(n)}(0)$, where $g(x) = (1+x^2)^{-1}$.

[5 marks]

12. In each of the following cases, calculate the radius of convergence R of each of the power series, and determine whether the series converges at $x = \pm R$.

a) $\sum_{n=0}^{\infty}(n+1)x^n$; b) $\sum_{n=0}^{\infty}\frac{1}{4^n(n+1)}x^n$.

[15 marks]

13. Use calculus to show that the function $f(x) = x^3+2x^2+x-2$ has exactly one zero, and that this zero is in (0,1).

[6 marks]

Use the Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

with an initial guess $x_0 = 1$ to obtain successive approximations x_1, x_2, and x_3 to the solution of the equation $f(x) = 0$ in $(0,1)$. Give x_2, x_3 to 8 decimal places, and $f(x_3)$ to 1 significant figure.

[9 marks]

14. Let $f(x)$ be defined by

$$f(x) = \begin{cases} (2-x)^{-1} & \text{if } x \in (-\infty, 0], \\ x + \frac{1}{2} & \text{if } x \in (0, 2], \\ x + (2-x)^{-1} & \text{if } x \in (2, \infty). \end{cases}$$

Find any horizontal and vertical asymptotes of f. Identify any points of discontinuity of f. Calculate $f'(x)$ where it exists. Identify any points where f is not differentiable. Show that f has no stationary points, and determine maximal intervals on which f is decreasing and maximal intervals on which f is increasing. Find any zeros of f. Sketch the graph of f.

[15 marks]

15.

a) Compute $(1 + j)^{27}$.

b) Find all z in the form $a + bj$, for real a and b, with $z^4 = -16$.

Hint: You might find de Moivre's Theorem useful in both parts. [15 marks]