Math191 Class Test 2009— Solutions

1. State the domain and range of the following functions:

a) $f(x) = \cos(2x)$;

Solution The domain is $\mathbb{R}$ and the range is $[-1, 1]$. [3 marks]

b) $g(x) = |x + 2|$.

Solution The domain is $\mathbb{R}$ and the range is $[0, \infty)$ [3 marks]

2. Let

$$f(x) = \frac{2x+3}{x-1}.$$

Find the inverse function $f^{-1}(x)$.

Solution. Set $y = f(x)$ and solve for x in terms of y:

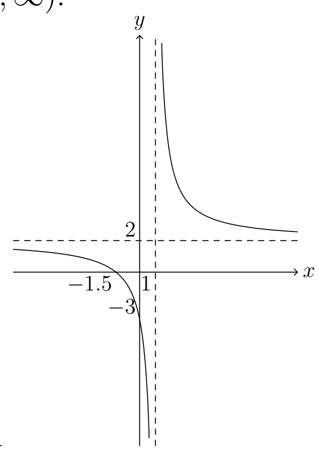
$$y = \frac{2x+3}{x-1} \quad \Leftrightarrow \quad (x-1)y = 2x+3 \Leftrightarrow x(y-2) = y+3 \quad \Leftrightarrow \quad x = \frac{y+3}{y-2}$$

So

$$f^{-1}(x) = \frac{x+3}{x-2}$$

[3 marks]

The domain of f is $(-\infty, 1) \cup (1, \infty)$. The range of f is the domain of f^{-1}, that is, $(-\infty, 2) \cup (2, \infty)$. [3 marks]



The graph is as shown

[4 marks]

3.

a) Find the exact value of $\cos^{-1}(-1/2)$.

Solution. $\cos^{-1}(-1/2) = 2\pi/3$. [2 marks] Give the general solution of the equation

$$\cos x = -\frac{1}{2}.$$

Solution.

b) The general solution to $\cos x = -1/2$ is

$$x = \pm\cos^{-1}(-1/2) + 2n\pi = \pm\frac{2\pi}{3} + 2n\pi$$

for $n \in \mathbb{Z}$. [4 marks]

4. In this question,give exact answers (in terms of π, $\sqrt{3}$ etc.) and not for approximations to any number of decimal places.

a) Convert $(2, -2\sqrt{3})$ from Cartesian to polar coordinates.

Solution. $r = \sqrt{x^2 + y^2} = \sqrt{4 + 12} = 4$ and $\tan\theta = y/x = -2\sqrt{3}/2 = -\sqrt{3}$. Hence

$$\theta = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3} + n\pi$$

for some $n \in \mathbf{Z}$. Since $(2, -2\sqrt{3})$ is in the right half-plane, we may take $n = 0$ and $\theta = -\pi/3$. [4 marks]

b) Convert $(1, 7\pi/6)$ from polar to Cartesian coordinates.

Solution.

$$\begin{aligned} x = r\cos\theta &= \cos(7\pi/6) = -\frac{\sqrt{3}}{2} \\ y = r\sin\theta &= \sin(7\pi/6) = -\frac{1}{2}. \end{aligned}$$

[2 marks]

5. Determine whether the following limits exist. Where they exist, evaluate them.

a) $\lim_{x\to\infty} \frac{x+1}{x-1}$

Solution.

$$\lim_{x\to\infty} \frac{x+1}{x-1} = \lim_{x\to\infty} \frac{1+\frac{1}{x}}{1-\frac{1}{x}} = 1.$$

[3 marks]

b) $\lim_{x\to-1} \frac{x^2-1}{x^3+1}$

Solution.

$$\lim_{x\to-1} \frac{x^2-1}{x^3+1} = \lim_{x\to-1} \frac{(x-1)(x+1)}{(x+1)(x^2-x+1)} = \lim_{x\to-1} \frac{x-1}{x^2-x+1} = -\frac{2}{3}.$$

Alternatively since $x^2-1 = 0 = x^3+1$ at $x=-1$ we can apply l'Hopital's Rule, and we obtain

$$\lim_{x\to-1} \frac{x^2-1}{x^3+1} = \lim_{x\to-1} \frac{2x}{3x^2} = -\frac{2}{3}.$$

[3 marks]

6

Differentiate the following functions. In part a), also find the tangent line through the point $(1,0)$.

a) $f(x) = x^3 - 1$

Solution. $f'(x) = 3x^2$. The tangent line is $y = f'(1)(x-1)$, that is, $y = 3(x-1) = 3x - 3$.

[4 marks]

b) $f(x) = x^2 \cos(2x-1)$

Solution. Using the Product Rule and the Chain Rule, $f'(x) = 2x\cos(2x-1) - 2x^2\sin(2x-1)$.

[3 marks]

c) $f(x) = \dfrac{2x}{x^2+1}$

Solution. Using the Quotient Rule, $f'(x) = \dfrac{2(x^2+1) - (2x \times 2x)}{(x^2+1)^2} = \dfrac{2-2x^2}{(x^2+1)^2}$.

[4 marks]

7

a) Find the Maclaurin series of $f(x) = (1-x)^{-1}$

Solution

$$f'(x) = (1-x)^{-2}, \quad f''(x) = 2!(1-x)^{-3}, \cdots f^{(n)}(x) = n!(1-x){-n-1}$$

So

$$f(0) = 1, \quad f'(0) = 1, \quad \frac{f''(0)}{2!} = 1 \cdots \frac{f^{(n)}(0)}{n!} = 1$$

So the Maclaurin series of f is

$$1 + x + x^2 \cdot + x^n \cdots = \sum_{n=0}^{\infty} x^n$$

[4 marks]

b) Hence, or otherwise, find the Maclaurin series of $g(x) = (1-2x^2)^{-1}$. **Solution** This is obtained by replacing x by $2x^2$ in the Maclaurin series for f. So the Maclaurin series of g is

$$1 + 2x^2 + 4x^4 + \cdots + 2^n x^{2n} \cdots = \sum_{n=0}^{\infty} 2^n x^{2n}$$

[2 marks]