Chapter 3

Integration (F.11)

3.1 The area under a curve (F.11.26–32)

Suppose that $f(x)$ is a continuous function between $x = a$ and $x = b$ (where $a \leq b$). We write

$$\int_a^b f(x)dx$$

for the area under the graph $y = f(x)$ between a and b: the 'integral of $f(x)$ between a and b'.

Examples

a) Let $f(x) = 2$. Then $\int_1^4 f(x)dx = 6$.

b) Let $f(x) = x$. Then $\int_0^6 f(x)dx = 18$.

By convention, areas underneath the x-axis are taken to be negative: thus, for example $\int_{-\pi}^{\pi} \sin(x)dx = 0$. Also, if $b < a$ then we let

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

Thus, for example,

$$\int_6^0 xdx = -18.$$

The first aim of integration is to calculate such areas for as many functions $f(x)$ as possible.

Given $f(x)$, let $F(x) = \int_0^x f(x)dx$, the area under the graph between 0 and x. Notice that $F(x)$ is itself a function.

Examples

a) Let $f(x) = 2$. Then $F(x) = \int_0^x 2dx = 2x$.

b) Let $f(x) = x$. Then $F(x) = \int_0^x xdx = x^2/2$.

Notice that in each case, $F'(x) = f(x)$. This is always the case: for

$$\begin{aligned} F'(x) &= \lim_{\delta \to 0} \frac{F(x+\delta) - F(x)}{\delta} \\ &= \lim_{\delta \to 0} \frac{\int_0^{x+\delta} f(x)dx - \int_0^x f(x)dx}{\delta} \\ &= \lim_{\delta \to 0} \frac{\int_x^{x+\delta} f(x)dx}{\delta} \qquad \text{(pic)}. \end{aligned}$$

But the closer δ gets to 0, the closer $\int_x^{x+\delta} f(x)dx$ gets to $f(x)\delta$, and hence the closer

$$\frac{\int_x^{x+\delta} f(x)dx}{\delta}$$

gets to $f(x)$. Hence $F'(x) = f(x)$.

Thus $F(x)$ is a function whose derivative is $f(x)$: this is the *fundamental theorem of calculus.*

Now $\int_a^b f(x)dx = F(b) - F(a)$ (picture): thus we have:

$$\int_a^b f(x)dx = F(b) - F(a),$$

where $F(x)$ is a function whose derivative is $f(x)$. $F(b) - F(a)$ is commonly written $[F(x)]_a^b$.

Examples

a) Let $f(x) = x^2$. A function whose derivative is $f(x)$ is $F(x) = x^3/3$. Hence

$$\int_1^3 x^2 dx = \left[\frac{x^3}{3}\right]_1^3 = \frac{3^3}{3} - \frac{1^3}{3} = 9 - \frac{1}{3} = 8\frac{2}{3}.$$

b) Let $f(x) = \cos x$. A function whose derivative is $f(x)$ is $F(x) = \sin x$. Hence

$$\int_0^{\pi/2} \cos x dx = [\sin x]_0^{\pi/2} = \sin(\pi/2) - \sin(0) = 1 - 0 = 1.$$

3.2 Definite and indefinite integrals

Notice that if the derivative of $F(x)$ is $f(x)$, then so is the derivative of $F(x)+C$ for all constants C. Hence we have a choice of functions $F(x)$ with $F'(x) = f(x)$.

When we're working out $\int_a^b f(x)dx$, this choice doesn't matter, since

$$[F(x) + C]_a^b = F(b) + C - (F(a) + C) = F(b) - F(a) = [F(x)]_a^b$$

.

$\int_a^b f(x)dx$ is called a *definite* integral: it has a definite value (the area under the curve from a to b. That is, the definite integral is a *number*.

Sometimes we don't want to specify a and b: this gives an *indefinite* integral, in which we can choose a constant. For example

$$\int x^2 dx = \frac{x^3}{3} + C,$$

where C is a constant. This is the *indefinite integral of* x^2. The indefinite integral is a *function*, with an *arbitrary constant* C.

3.3 Common integrals (F.11.1–15)

We can integrate a lot of functions just by spotting what they're the derivative of. For example, we know $\frac{d}{dx}x^{a+1} = (a+1)x^a$, so

$$\int x^a dx = \frac{x^{a+1}}{a+1} + C.$$

This formula is fine provided $a \neq -1$. To integrate $x^{-1} = 1/x$, remember that $\frac{d}{dx}\ln x = 1/x$, so

$$\int \frac{1}{x} dx = \ln x + C \qquad (x > 0).$$

If $x < 0$, then $\frac{d}{dx}\ln(-x) = \frac{-1}{-x} = \frac{1}{x}$, so

$$\int \frac{1}{x} dx = \ln(-x) + C \qquad (x < 0).$$

Combining these, we get

$$\int \frac{1}{x} dx = \ln(|x|) + C \qquad (x \neq 0).$$

Similarly we have

$$\begin{aligned}
\int \sin x dx &= -\cos x + C \\
\int \cos x dx &= \sin x + C \\
\int e^x dx &= e^x + C \\
\int (ax+b)^k dx &= \frac{1}{a(k+1)}(ax+b)^{k+1} \qquad (k \neq -1) \\
\int \sin(ax+b) dx &= \frac{-1}{a}\cos(ax+b) + C \\
\int \cos(ax+b) dx &= \frac{1}{a}\sin(ax+b) + C \\
\int e^{ax+b} &= \frac{1}{a}e^{ax+b} + C.
\end{aligned}$$

See handout of common integrals. Note: using the rules of differentiation, we can differentiate almost any function. The same is not true of integration: e.g. there is no good expression for $\int e^{x^2} dx$.

Examples

a)

$$\int_1^4 2e^x + \frac{1}{x} dx = [2e^x + \ln(|x|)]_1^4 = 2e^4 + \ln(4) - (2e^1 + \ln(1)) = 105.146\ldots$$

b)

$$\int_0^\pi \cos(2x) + \sin x dx = [\frac{\sin(2x)}{2} - \cos x]_0^\pi = \frac{\sin(2\pi)}{2} - \cos(\pi) - (\frac{\sin(0)}{2} - \cos 0) = 0 - (-1) - (0-1) = 2.$$

c)

$$\int_1^2 \sqrt{2x+1} dx = \int_1^2 (2x+1)^{1/2} dx = [\frac{1}{3}(2x+1)^{3/2}]_1^2 = \frac{1}{3}(5^{3/2} - 3^{3/2}) = 1.9947\ldots.$$