Chapter 5

Complex numbers ($\mathbb{C}$)

5.1 Historical motivation (3.1)

Consider the equation $z^2 - z + 1 = 0$. Using the formula $(-b \pm \sqrt{b^2 - 4ac})/2a$ we get

$$z = \frac{1 \pm \sqrt{-3}}{2}.$$

No solutions? If we write $j = \sqrt{-1}$, then we have

$$z = \frac{1 \pm \sqrt{(3)(-1)}}{2} = \frac{1 \pm \sqrt{3}\sqrt{-1}}{2} = \frac{1 \pm \sqrt{3}j}{2}.$$

There are two solutions, but they are *complex numbers*: they are of the form $x + yj$, where x and y are real numbers, and $j = \sqrt{-1}$. NB j is often written j.

The real numbers are *not algebraically closed*: you can write down a polynomial equation, using only real numbers, which has no real solutions. The complex numbers are *algebraically closed*: any polynomial equation (even involving complex numbers) has a complex solution. This is one of many mathematical justifications for considering complex numbers.

In fact, the justifications are not just mathematical: in a deep sense, complex numbers are the fundamental number system of the universe, and it is our own limitations which cause us to see them as less natural than real numbers (e.g. quantum mechanics). Even in relatively straightforward physical situations, calculations can become much easier and more natural if we use complex numbers rather than real numbers (e.g. complex impedance, section 3.5).

5.2 Basic definitions and properties (3.2)

A complex number z is one of the form $z = x + yj$, where x and y are real numbers, and $j^2 = -1$. The *set* of all complex numbers is written $\mathbb{C}$.

A complex number z has a *real part* $x = \Re(z)$, and an imaginary part $y = \Im(z)$. For example, if $z = 2 - 3j$, then $\Re(z) = 2$ and $\Im(z) = -3$ (Note: the imaginary part doesn't include the j). If $\Im(z) = 0$ then z is a real number. If $\Re(z) = 0$ then z is called an *imaginary* number — not a very useful term.

Two complex numbers are equal if their real and imaginary parts are equal: i.e. $a + bj = c + dj$ if $a = c$ and $b = d$.

If $z = a + bj$ is a complex number, its *complex conjugate* $\bar{z}$ is the complex number $\bar{z} = a - bj$ obtained by changing the sign of the imaginary part. Note also written z^*. We have $z = \bar{z}$ if and only if z is a real number.

Since a complex number $z = x + jy$ is described by two real numbers x and y, we can represent it by the point (x, y) in the plane. Draw and show some examples. Real axis and imaginary axis. Note that $\bar{z}$ is the reflection of z in the real axis.

Arithmetic with complex numbers

Addition and subtraction work just as with vectors: if $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$ then $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$ and $z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$. Examples: $(2 + j) + (3 - 2j) = 5 - j$, $(2 - 2j) - (3 - j) = -1 - j$.

Notice that if $z = x + jy$, then $z + \bar{z} = (x + jy) + (x - jy) = 2x$. Thus $z + \bar{z} = 2\Re(z)$. Similarly, $z - \bar{z} = (x + jy) - (x - jy) = 2jy$. Thus $z - \bar{z} = 2j\Im(z)$.

For multiplication, we multiply them out the two expressions and remember that $j^2 = -1$. Thus $(2 + j)(3 - 2j) = 6 - 2j^2 + 3j - 4j = 8 - j$, and $(2 - 2j)(3 - j) = (6 + 2j^2 - 6j - 2j = 4 - 8j$. In general $(x_1 + jy_1)(x_2 + jy_2) = (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)j$.

Note $j^2 = -1$, $j^3 = -j$, $j^4 = 1$, $j^5 = j$, $j^6 = -1$, $j^7 = -j$, $j^8 = 1$, etc.

An important fact is that, for any complex number z, $z\bar{z}$ is always a real number. For if $z = x + jy$, then $\bar{z} = x - jy$, and

$$z\bar{z} = (x + jy)(x - jy) = x^2 - j^2y^2 + jxy - jxy = x^2 + y^2.$$

Thus for example $(2 + j)(\overline{2 + j}) = 2^2 + 1^2 = 5$. Moreover, $z\bar{z} = x^2 + y^2 = r^2$, where r is the distance of z from the origin in the Argand diagram. We write this distance as $|z|$, the *modulus* of z: thus

$$z\bar{z} = |z|^2 \qquad \text{or } |z| = \sqrt{z\bar{z}}.$$

We can use this to divide two complex numbers z_1 and z_2: if $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$, then we simplify

$$\frac{z_1}{z_2} = \frac{x_1 + jy_1}{x_2 + jy_2}$$

by multiplying top and bottom by $\bar{z}_2 = x_2 - jy_2$: this makes the bottom into a real number, which we can just divide by.

Examples

a)

$$\frac{1}{j} = \frac{-j}{j(-j)} = -j \qquad \text{(important)}.$$

b)

$$\frac{1+j}{1-j} = \frac{(1+j)(1+j)}{(1-j)(1+j)} = \frac{2j}{1^2+1^2} = j.$$

(You can check $j(1-j) = j - j^2 = 1 + j$).

c)

$$\frac{2+j}{3-2j} = \frac{(2+j)(3+2j)}{(3-2j)(3+2j)} = \frac{4+7j}{3^2+2^2} = \frac{4}{13} + \frac{7}{13}j.$$

5.3 The polar form of complex numbers (3.2.5,3.2.6)

Just as with points (x, y), complex numbers can be represented in polar coordinates: we can describe a complex number $z = x + jy$ by its distance r from the origin, and its angle θ with the origin.

We've already seen that $r = |z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$, the *modulus* of z. θ is given by $\tan\theta = y/x$, and is called the *argument* of z, written $\arg(z)$.

Remember (from polar coordinates) that $\tan^{-1}(y/x)$ is an angle between $-\pi/2$ and $\pi/2$. Thus we can't just say $\arg(z) = \tan^{-1}(y/x)$: we have to look at where z is in the argand diagram. In fact, it is traditional to give $\arg(z)$ as an angle between 0 and 2π: thus we have

$$\arg(z) = \begin{cases} \tan^{-1}(y/x) & \text{if } x > 0, y \geq 0, \\ \frac{\pi}{2} & \text{if } x = 0, y > 0, \\ \tan^{-1}(y/x) + \pi & \text{if } x < 0, \\ \frac{3\pi}{2} & \text{if } x = 0, y < 0, \\ \tan^{-1}(y/x) + 2\pi & \text{if } x > 0, y < 0. \end{cases}$$

Examples $\arg(1) = 0$, $\arg(1+j) = \pi/4$, $\arg(j) = \pi/2$, $\arg(-1+j) = 3\pi/4$, $\arg(-1) = \pi$, $\arg(-1-j) = 5\pi/4$, $\arg(-j) = 3\pi/2$, $\arg(1-j) = 7\pi/4$.

If $z = x + jy$ and $|z| = r$, $\arg(z) = \theta$ then (draw picture) $x = r\cos\theta$ and $y = r\sin\theta$. Thus $z = r\cos\theta + jr\sin\theta$, or

$$z = r(\cos\theta + j\sin\theta),$$

In fact, we can write this in a better way. Recall

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Thus

$$e^{j\theta} = 1 + (j\theta) + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \cdots$$

Now $j^2 = -1$, $j^3 = -j$, $j^4 = 1$, $j^5 = j$ etc., so

$$\begin{aligned} e^{j\theta} &= 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} - \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos\theta + j\sin\theta. \end{aligned}$$

This is *Euler's formula*: $e^{j\theta} = \cos\theta + j\sin\theta$.

Thus

$$z = re^{j\theta},$$

the *polar form* of z.

Examples Express the following complex numbers in polar form:

a) $z = 1 + j$.

We have $|z| = \sqrt{2}$ and $\arg(z) = \tan^{-1}(1) = \pi/4$, so

$$1 + j = \sqrt{2}e^{\frac{j\pi}{4}}.$$

b) $z = -1$.

We have $|z| = 1$ and $\arg(z) = \pi$, so

$$-1 = e^{j\pi}.$$

c) $z = -2 - 3j$.

We have $|z| = \sqrt{2^2 + 3^2} = \sqrt{13}$, and $\arg(z) = \tan^{-1}(-3/-2) + \pi \simeq 4.646$, so

$$-2 - 3j = \sqrt{13}e^{4.646j}.$$

In polar coordinates, multiplication and division of complex numbers becomes very easy (while addition and subtraction become harder).

If $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$, then

$$z_1 z_2 = r_1 e^{j\theta_1} r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1+\theta_2)}.$$

(multiply the moduli and add the arguments).

$$\frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)}.$$

(divide the moduli and subtract the arguments).

Remarks on Euler's formula $e^{j\theta} = \cos\theta + j\sin\theta$

Fjrst, it is important to recognize $e^{j\theta}$ as the complex number which is distance 1 from the origin at angle θ (draw picture with circle of radius 1 and a few examples on it).

Since $\cos\theta$ and $\sin\theta$ are unchanged when you add 2π to θ, so is $e^{j\theta}$. Thus

$$1 = e^{0j} = e^{2\pi j} = e^{4\pi j} = e^{6\pi j} = e^{-2\pi j} = e^{-4\pi j} = \cdots$$

$$j = e^{\pi j/2} = e^{5\pi j/2} = e^{9\pi j/2} = e^{-3\pi j/2} = \cdots$$

and so on.

$$e^{-j\theta} = \cos(-\theta) + j\sin(-\theta) = \cos\theta - j\sin\theta = \overline{\cos\theta + j\sin\theta}.$$

Thus the complex conjugate of $e^{j\theta}$ is

$$\overline{e^{j\theta}} = e^{-j\theta}.$$

(Picture).

Adding the equations

$$\begin{aligned} e^{j\theta} &= \cos\theta + j\sin\theta \\ e^{-j\theta} &= \cos\theta - j\sin\theta \end{aligned}$$

we get

$$e^{j\theta} + e^{-j\theta} = 2\cos\theta$$

or

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}.$$

Subtracting them gives

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}.$$

This shows the relationship between the trigonometric functions cos and sin, and the hyperbolic functions cosh and sinh. In fact,

$$\cos\theta = \cosh(j\theta),$$

$$\sin\theta = \frac{1}{j}\sinh(j\theta) = -j\sinh(j\theta).$$

So, from the point of view of complex numbers cos and cosh are essentially the same function, as are sin and sinh. This also explains Osborn's rule: every time we have a product of two sines, we get $(-j)^2 = -1$.

5.4 de Moivre's theorem (3.3.1, 3.3.2)

We have $e^{j\theta} = \cos\theta + j\sin\theta$. Hence

$$(\cos\theta + j\sin\theta)^n = (e^{j\theta})^n = e^{jn\theta} = \cos(n\theta) + j\sin(n\theta).$$

That is

$$(\cos\theta + j\sin\theta)^n = \cos(n\theta) + j\sin(n\theta).$$

This is *de Moivre's theorem.* One of its uses is to obtain trigonometric identites easily. These are of two main types:

First, we can write $\cos(n\theta)$ and $\sin(n\theta)$ in terms of $\cos\theta$ and $\sin\theta$. The method is to write

$$\cos(n\theta) + j\sin(n\theta) = (\cos\theta + j\sin\theta)^n,$$

and expand the right hand side using the binomial theorem.

Examples

a)

$$\begin{aligned}\cos(2\theta) + j\sin(2\theta) &= (\cos\theta + j\sin\theta)^2 \\ &= \cos^2\theta - \sin^2\theta + 2j\sin\theta\cos\theta.\end{aligned}$$

Equating the real parts gives $\cos(2\theta) = \cos^2\theta - \sin^2\theta$, equating imaginary parts gives $\sin(2\theta) = 2\sin\theta\cos\theta$.

b) With more complicated examples, it's usual to abbreviate $c = \cos\theta$ and $s = \sin\theta$.

$$\begin{aligned}\cos(3\theta) + j\sin(3\theta) = (c + js)^3 \\ &= c^3 + 3c^2(js) + 3c(js)^2 + (js)^3 \\ &= c^3 + 3jc^2s - 3cs^2 - js^3 \\ &= (c^3 - 3cs^2) + j(3c^2s - s^3).\end{aligned}$$

Thus $\cos(3\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta$, and $\sin(3\theta) = 3\cos^2\theta\sin\theta - \sin^3\theta$.

c)

$$\begin{aligned}\cos(5\theta) + j\sin(5\theta) = (c + js)^5 \\ &= c^5 + 5c^4(js) + 10c^3(js)^2 + 10c^2(js)^3 + 5c(js)^4 + (js)^5 \\ &= (c^5 - 10c^3s^2 + 5cs^4) + j(5c^4s - 10c^2s^3 + s^5).\end{aligned}$$

Thus $\cos(5\theta) = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$, and $\sin(5\theta) = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$.

Second we can write powers of $\cos\theta$ and $\sin\theta$ in terms of $\cos(n\theta)$ and $\sin(n\theta)$. To do this, write $z = \cos\theta + j\sin\theta$, so

$$z^n = \cos n\theta + j\sin n\theta \tag{5.1}$$

$$z^{-n} = \cos n\theta - j\sin n\theta. \tag{5.2}$$

Adding gives

$$2\cos n\theta = z^n + z^{-n},$$

and subtracting gives

$$2j\sin n\theta = z^n - z^{-n}.$$

To get an expression for $\cos^k\theta$, we write $2\cos\theta = z + z^{-1}$, so

$$2^n\cos^n\theta = \left(z + z^{-1}\right)^n.$$

Expand this using the binomial theorem, and use (1) and (2) to simplify. Similarly for $\sin^k \theta$, using $2j \sin\theta = z - z^{-1}$.

Examples

a)

$$\begin{aligned} 2^3 \cos^3 \theta &= (z + z^{-1})^3 \\ &= z^3 + 3z^2 z^{-1} + 3zz^{-2} + z^{-3} \\ &= (z^3 + z^{-3}) + 3(z + z^{-1}) \\ &= 2\cos 3\theta + 6\cos\theta. \end{aligned}$$

So

$$4\cos^3\theta = \cos 3\theta + 3\cos\theta.$$

b)

$$\begin{aligned} (2j)^3 \sin^3 \theta &= (z - z^{-1})^4 \\ &= z^3 - 3z + 3z^{-1} - z^{-3} \\ &= (z^3 - z^{-3}) - 3(z - z^{-1}) \\ &= 2j\sin 3\theta - 6j\sin\theta. \end{aligned}$$

Thus

$$8j^3 \sin^3\theta = 2j\sin 3\theta - 6j\sin\theta,$$

or

$$4\sin^3\theta = -\sin 3\theta + 3\sin\theta.$$

c)

$$\begin{aligned} (2j)^4 \sin^4 \theta &= (z - z^{-1})^4 \\ &= z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4} \\ &= 2\cos 4\theta - 8\cos 2\theta + 6. \end{aligned}$$

Thus

$$8\sin^4\theta = \cos 4\theta - 4\cos 2\theta + 3.$$

5.5 Roots of complex numbers

De Moivre's Theorem can be used to find the square roots of a complex number, or the cube roots, or the k'th roots, for any integer k. Any nonzero complex number z has k different k'th roots. If we write s in polar form, $z = re^{j\theta}$ then the k'th roots are given by $r^{1/k}e^{j(\theta+2\pi m)/k}$ for $0 \leq k \leq m$.

Examples

a) If $z = 1 + j$ then $= \sqrt{2}e^{j\pi/4}$. Then the square roots of z are

$$2^{1/4}e^{j\pi/8} = 2^{1/4}(\cos(\pi/8) + j\sin(\pi/8))$$

and

$$2^{1/4}e^{j9\pi/8} = 2^{1/4}(\cos(9\pi/8)+j\sin(9\pi/8)) = -2^{1/4}e^{j\pi/8} = -2^{1/4}(\cos(\pi/8)+j\sin(\pi/8)).$$

b) If $z = -8j$, then $z = 8e^{j(3\pi/2)}$. The cube roots are then $2e^{j(\pi/2+2\pi m/3)}$ for $m = 0$, 1 2. So the cube roots are

$$2e^{j\pi/2} = 2j, \quad 2e^{j(\pi/2+2\pi/3} = 2e^{j7\pi/6} = -\sqrt{3} - j,$$

$$2e^{j(\pi/2+4\pi/3} = 2e^{j11\pi/6} = \sqrt{3} - j.$$