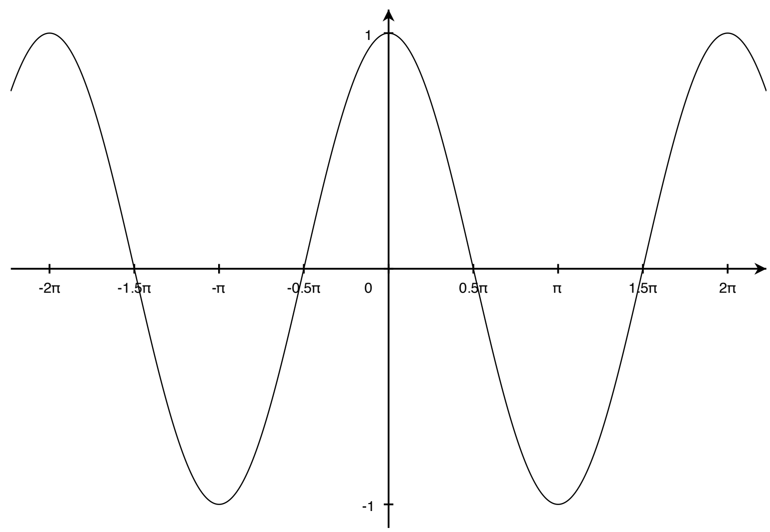
1. The maximal domain is $\mathbf{R}$ and the range is $[-1, 1]$ (1 mark each).

   The graph is shown below (1 mark). It crosses the x-axis at $x = (2n+1)\pi/2$ for $n \in \mathbf{Z}$ and the y-axis at $y = 1$ (1 mark).



2. We have $f(0) = 0$, $f'(x) = \frac{3}{3x+1}$, so $f'(0) = 3$, and $f''(x) = -\frac{9}{(1+3x)^2}$, so $f''(0) = -9$. (1 mark each for $f(0)$, $f'(0)$, and $f''(0)$).

   Hence the first three terms in the Maclaurin series expansion of $f(x)$ are

$$f(x) = 3x - \frac{9}{2}x^2 + \cdots.$$

   (1 mark for correct coefficients carried forward from $f(0)$, $f'(0)$, and $f''(0)$. 1 mark for not saying $f(x) = 3x - \frac{9}{2}x^2$).

3. (a) $r = \sqrt{4+4} = \sqrt{8}$ (1 mark) and $\tan\theta = 2/-2 = -1$, so since $x < 0$ we have $\theta = \tan^{-1}(-1) + \pi = -\pi/4 + \pi = 3\pi/4$ (3 marks).

   (b) $x = 2\cos(\pi/3) = 1$ and $y = 2\sin(\pi/3) = \sqrt{3}$. (1 mark each)

   Subtract one mark for each answer not given exactly.

4.

$$\begin{aligned} \int_1^4 (\sinh x + x^2)\, dx &= \left[\cosh x + \frac{1}{3}x^3\right]_1^4 \qquad \text{(3 marks)} \\ &= \left(\cosh 4 + \frac{64}{3}\right) - \left(\cosh 1 + \frac{1}{3}\right) \\ &= 46.765 \qquad \text{(2 marks)} \end{aligned}$$

   to three decimal places.

5. Differentiating the equation with respect to x gives

$$3x^2 + 2xy + x^2\frac{dy}{dx} - 3y^2 - 6xy\frac{dy}{dx} = 0 \quad \text{(3 marks)}.$$

Hence

$$\frac{dy}{dx} = \frac{3y^2 - 2xy - 3x^2}{x^2 - 6xy} \quad \text{(2 marks)}.$$

Thus $\frac{dy}{dx}$ is equal to 2/5 when $(x, y) = (1, 1)$. (1 mark).

The equation of the tangent at this point is therefore

$$y = \frac{2}{5}(x-1) + 1 = \frac{2}{5}x + \frac{3}{5} \qquad \text{(2 marks)}.$$

6. (a) By the product rule,

$$\frac{d}{dx}(x^3 \sinh x) = 3x^2 \sinh x + x^3 \cosh x. \qquad \text{(2 marks)}.$$

(b) By the chain rule,

$$\begin{aligned} \frac{d}{dx}\left((1+\cos x)^5\right) &= 5(1+\cos x)^4(-\sin x) \\ &= -5\sin x(1+\cos x)^4 \qquad \text{(2 marks)}. \end{aligned}$$

(c) By the quotient rule,

$$\begin{aligned} \frac{d}{dx}\left(\frac{x^3}{2+\cos x}\right) &= \frac{3x^2(2+\cos x) - x^3(-\sin x)}{(2+\cos x)^2} \\ &= \frac{x^2(6 + 3\cos x + x\sin x)}{(2+\cos x)^2}. \qquad \text{(2 marks)}. \end{aligned}$$

7. Stationary points are given by solutions of $f'(x) = 0$, i.e. when

$$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3) = 0.$$

So there are two stationary points, namely $x = 1, 3$. (3 marks, 1 for the derivative and 2 for the solution.)

To determine their natures, $f''(x) = 6x - 12$ so $f''(1) < 0$ i.e. $x = 1$ is a local maximum and $f''(3) > 0$ i.e $x = 3$ is a local minimum. (2 marks, 1 for each stationary point).

8.

$$
\begin{aligned}
z_1 + z_2 &= 4 - j \qquad \text{(1 mark)} \\
z_1 - z_2 &= -2 + 3j \qquad \text{(1 mark)} \\
z_1 z_2 &= (1+j)(3-2j) = 3 + 3j - 2j - 2j^2 = 5 + j \qquad \text{(2 marks)} \\
z_1/z_2 &= \frac{1+j}{3-2j} = \frac{(1+j)(3+2j)}{(3-2j)(3+2j)} = \frac{1+5j}{13} \qquad \text{(2 marks).}
\end{aligned}
$$

9. $\cos^{-1}(1/\sqrt{2}) = \pi/4$ (1 mark). The general solution of $\cos\theta = 1/\sqrt{2}$ is

$$\theta = \pm\pi/4 + 2n\pi \qquad (n \in \mathbf{Z}) \qquad \text{(3 marks).}$$

10.

$$
\begin{aligned}
\mathbf{a} + \mathbf{b} &= \mathbf{i} + 4\mathbf{j} + 2\mathbf{k} \quad \text{(1 mark)} \\
\mathbf{a} - \mathbf{b} &= -3\mathbf{i} + 2\mathbf{j} \quad \text{(1 mark)} \\
|\mathbf{a}| &= \sqrt{1^2 + 3^2 + 1^2} = \sqrt{11} \qquad \text{(1 mark)} \\
|\mathbf{b}| &= \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6} \qquad \text{(1 mark)} \\
\mathbf{a} \cdot \mathbf{b} &= -2 + 3 + 1 = 2 \qquad \text{(1 mark).}
\end{aligned}
$$

Hence the angle θ between $\mathbf{a}$ and $\mathbf{b}$ is $\cos^{-1}(2/\sqrt{11}\sqrt{6}) = 1.322$ (1 mark).

11. The Maclaurin series expansion of $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \qquad \text{(2 marks)}$$

Hence

(a)

$$x^2 \cos x = x^2 - \frac{x^4}{2!} + \cdots \qquad \text{(2 marks)}$$

(b)

$$\cos(2x) = 1 - 2x^2 + \frac{2x^4}{3} - \cdots \qquad \text{(3 marks)}$$

(c)

$$(\cos x)^2 = (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots)^2 = 1 - x^2 + \frac{x^4}{3} - \cdots \qquad \text{(5 marks)}$$

We have

$$(\cos(0.1))^2 = 1 - \frac{0.01}{2} + \frac{0.0001}{3} - \cdots = 0.990033$$

to 6 decimal places. (3 marks)

12. The radius of the convergence R of the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

is given by

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

provided this limit exists. In this case $a_n = 1/(n^4\ 4^n)$, so

$$|a_n/a_{n+1}| = \frac{(n+1)^4\ 4^{n+1}}{n^4\ 4^n} = 4\left(\frac{n+1}{n}\right)^4,$$

which tends to 4 as $n \to \infty$. Hence $R = 4$. (8 marks).

When $x = -4$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^4}.$$

This converges by the alternating series test, which states that

$$\sum_{n=0}^{\infty} (-1)^n a_n$$

converges if a_n is a decreasing sequence with $a_n \to 0$. (3 marks)

When $x = 4$, the series becomes

$$\sum_{n=0}^{\infty} \frac{1}{n^4},$$

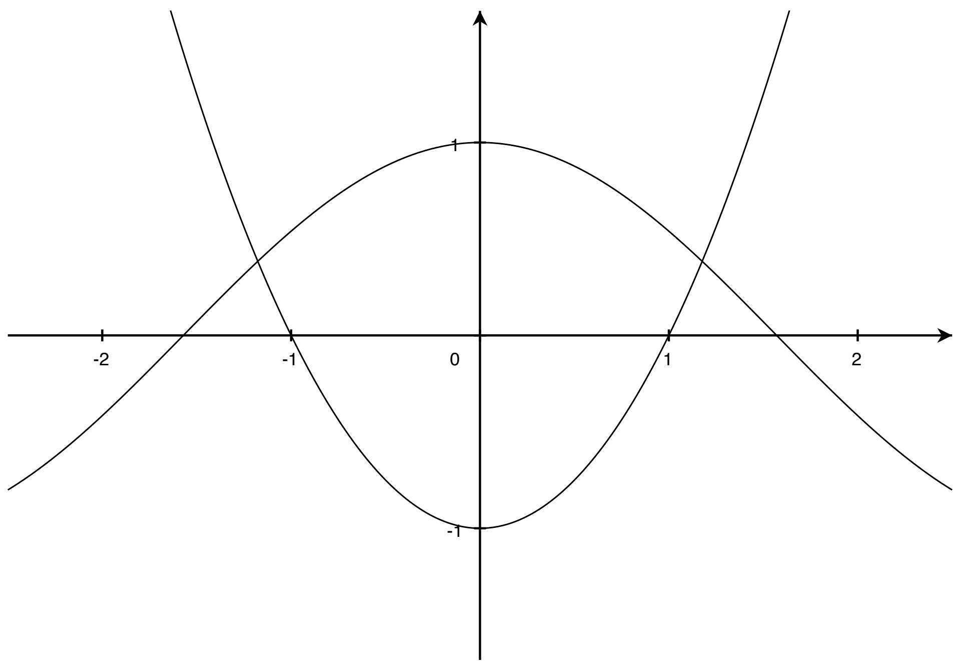
which converges by comparison with

$$\sum_{n=0}^{\infty} \frac{1}{n^2},$$

(whose convergence is a standard result). (3 marks).

Hence the series converges if and only if $-4 \le x \le 4$. (1 mark).

13. The graphs are as shown:



(5 marks).

Between 0 and 2 the function $x^2 - 1$ increases monotonically from -1 to 3 whereas $\cos x$ decreases monotonically from 1 to $\cos 2 < 0$. Hence they must cross once in this range. For $x > 2$ we have $x^2 - 1 > 3$ and $\cos x \leq 1$ so there are no further solutions. Since both functions are even there is also exactly one negative solution. (3 marks).

We have $f'(x) = 2x + \sin x$, so the Newton-Raphson formula becomes

$$x_{n+1} = x_n - \frac{x_n^2 - 1 - \cos x_n}{2x_n + \sin x_n} \quad \text{(3 marks)}.$$

Hence with $x_0 = 1.2$

$$x_1 = 1.176698.$$

$$x_2 = 1.176502.$$

$$x_3 = 1.176502.$$

(1 mark each).

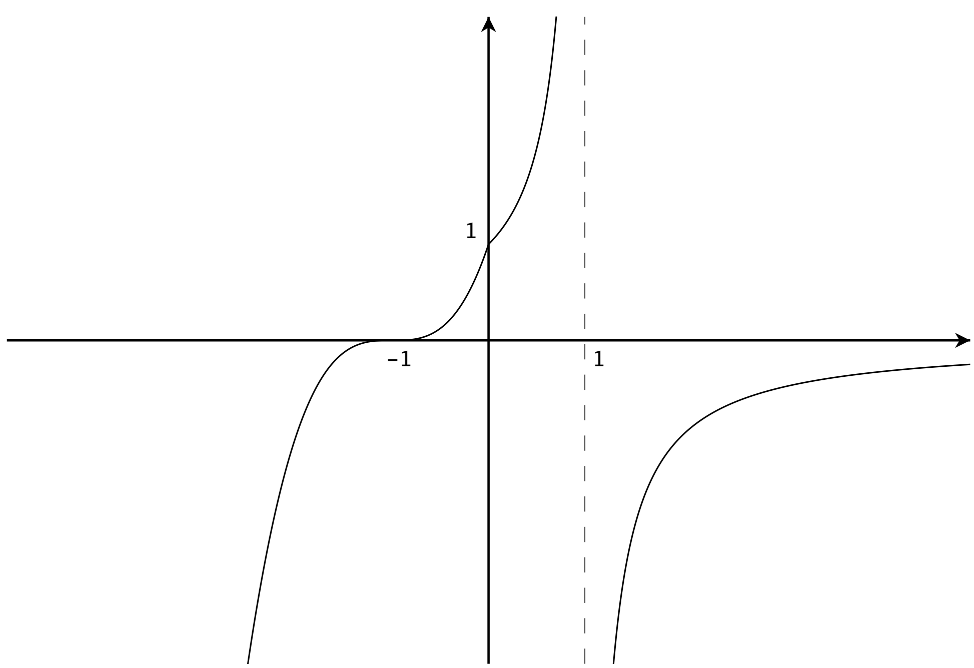
Since the functions are even the negative solution is $-\alpha$ where α is the positive solution. Hence -1.176502 is a good approximation (1 mark).

14. For $x \leq 0$ we have $f(x) = (x+1)^3$, which has a zero at $x = -1$. We have $f(0) = 1$.

The derivative is $f'(x) = 3(x+1)^2$, so $f(x)$ is increasing for $x \leq 0$ and there is a stationary point at $x = -1$ where $f(x) = 0$. Since $f''(x) = 6(x+1)$ is also zero at $x = -1$, we have to check the third derivative. It is $f'''(x) = 6 \neq 0$ so there is a point of inflection at $x = -1$. The gradient of $(x+1)^3$ at $x = 0$ is 3.

For $x > 0$ we have $f(x) = 1/(1-x)$, which has no zeros, tends to 1 as $x \to 0$, and tends to 0 as $x \to \infty$. We have $f'(x) = 1/(1-x)^2$, so there are no stationary points, and $f(x)$ is increasing in $(0,1) \cup (1,\infty)$. The gradient is 1 at $x = 0$. There is a vertical asymptote at $x = 1$.

The graph of $f(x)$ is therefore



(12 marks).

$f(x)$ is not continuous at $x = 1$, since 1 is not in its maximal domain. (1 mark).

$f(x)$ is not differentiable at $x = 1$ (not in maximal domain), nor at $x = 0$ (no well-defined tangent to the graph at this point). (2 marks).

15. By de Moivre's theorem

$$\begin{aligned}\cos 3\theta &= \mathrm{Re}(\cos\theta + j\sin\theta)^3 \\ &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ &= \cos^3\theta - 3\cos\theta(1-\cos^2\theta) \\ &= 4\cos^3\theta - 3\cos\theta \qquad \text{(5 marks).}\end{aligned}$$

So $a = 4$ and $b = -3$. Similarly

$$\begin{aligned}\sin 3\theta &= \mathrm{Im}(\cos\theta + j\sin\theta)^3 \\ &= 3\cos^2\theta\sin\theta - \sin^3\theta \\ &= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta \qquad \text{(5 marks).}\end{aligned}$$

So $c = -4$ and $d = 3$.

Substituting $\theta = \pi/6$ we obtain

$$\begin{aligned}\cos 3\theta &= \cos(\pi/2) = 0 \\ 4\cos^3\theta - 3\cos\theta &= 4(\cos(\pi/6))^3 - 3\cos(\pi/6) \\ &= 4(\sqrt{3}/2)^3 - 3\sqrt{3}/2 \\ &= \frac{3\sqrt{3}}{2} - \frac{3\sqrt{3}}{2} = 0 \qquad \text{(3 marks),}\end{aligned}$$

and

$$\begin{aligned}\sin 3\theta &= \sin(\pi/2) = 1 \\ 3\sin\theta - 4\sin^3\theta &= 3\sin(\pi/6) - 4(\sin(\pi/6))^3 \\ &= \frac{3}{2} - \frac{4}{2^3} \\ &= 1 \qquad \text{(2 marks).}\end{aligned}$$