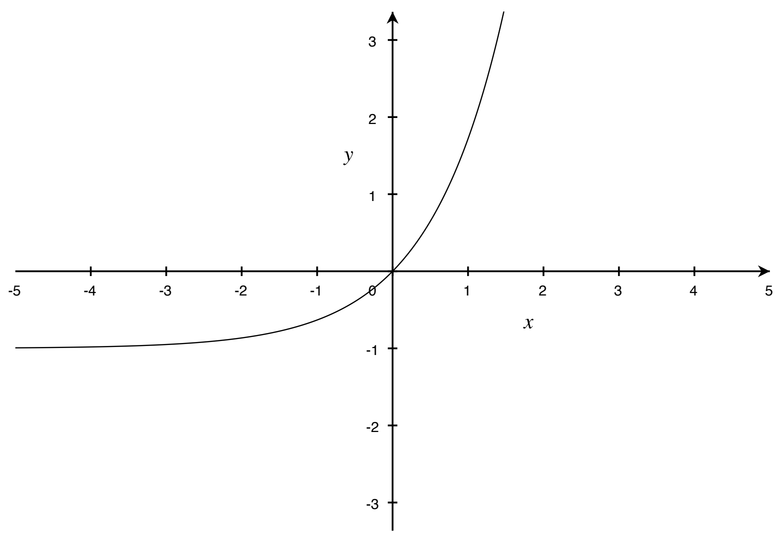
1. The maximal domain is $\mathbf{R}$ and the range is $(-1, \infty)$ (1 mark each).

   The graph is shown below (1 mark). It crosses the x and y-axes at $x = y = 0$ (1 mark).



2. We have $f(0) = 1$, $f'(x) = -\frac{3}{2}(1+3x)^{-3/2}$, so $f'(0) = -3/2$, and $f''(x) = \frac{27}{4}(1+3x)^{-5/2}$, so $f''(0) = 27/4$. (1 mark each for $f(0)$, $f'(0)$, and $f''(0)$).

   Hence the first three terms in the Maclaurin series expansion of $f(x)$ are

$$f(x) = 1 - 3x/2 + 27x^2/8 + \cdots.$$

   (1 mark for correct coefficients carried forward from $f(0)$, $f'(0)$, and $f''(0)$. 1 mark for not saying $f(x) = 1 - 3x/2 + 27x^2/8$).

3. (a) $r = \sqrt{1+1} = \sqrt{2}$ (1 mark). $\tan\theta = -1/1 = -1$, so since $x > 0$ we have $\theta = \tan^{-1}(-1) = -\pi/4$ (3 marks).

   (b) $x = 4\cos(-\pi/6) = 4(\sqrt{3}/2) = 2\sqrt{3}$ and $y = 4\sin(-\pi/6) = 4(-1/2) = -2$. (1 mark each)

   Subtract one mark for each answer not given exactly.

4.

$$\begin{aligned}\int_1^2 \frac{1}{x^2} + \cos x \, dx &= \left[-\frac{1}{x} + \sin x\right]_1^2 \qquad \text{(3 marks)} \\ &= (-\frac{1}{2} + \sin(2)) - (-1 + \sin(1)) \\ &= 0.568 \qquad \text{(2 marks)}\end{aligned}$$

   to three decimal places.

5. Differentiating the equation with respect to x gives

$$2x + 4y + 4x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \quad \text{(3 marks)}.$$

Hence

$$\frac{dy}{dx} = -\frac{2x+4y}{4x+2y} \quad \text{(2 marks)}.$$

Thus $\frac{dy}{dx}$ is equal to -1 when $(x, y) = (1, 1)$. (1 mark).
The equation of the tangent at this point is therefore

$$y = -(x-1) + 1 = 2 - x \qquad \text{(2 marks)}.$$

6. (a) By the product rule,

$$\frac{d}{dx}(1+x)e^{2x} = e^{2x} + 2(1+x)e^{2x} = (3+2x)e^{2x}. \qquad \text{(2 marks)}.$$

(b) By the chain rule,

$$\frac{d}{dx}(x^2+2x+5)^7 = 7(x^2+2x+5)^6(2x+2) = 14(x+1)(x^2+2x+5)^6 \qquad \text{(2 marks)}.$$

(c) By the quotient rule,

$$\frac{d}{dx}\left(\frac{\cos x}{1+x^4}\right) = \frac{-(1+x^4)\sin x - 4x^3\cos x}{(1+x^4)^2}. \qquad \text{(2 marks)}.$$

7. Stationary points are given by solutions of $f'(x) = 0$, i.e. when

$$f'(x) = -\frac{1}{x^2} + 2x = 0 \qquad \text{or} \qquad x^3 = \frac{1}{2},$$

so there is only one stationary point, namely $x = 2^{-1/3}$. (3 marks, 1 for the derivative and 2 for the solution.)

To determine its nature, $f''(x) = 2/x^3 + 2$ so $f''(2^{-1/3}) > 0$, and the stationary point is a local minimum. (2 marks, 1 for the second derivative and one for classifying the stationary point).

8.

$$\begin{aligned}
z_1 + z_2 &= 5 - j \qquad \text{(1 mark)}\\
z_1 - z_2 &= 3 + 3j \qquad \text{(1 mark)}\\
z_1 z_2 &= (4+j)(1-2j) = 4 - 8j + j - 2j^2 = 6 - 7j \qquad \text{(2 marks)}\\
z_1/z_2 &= \frac{4+j}{1-2j} = \frac{(4+j)(1+2j)}{(1-2j)(1+2j)} = \frac{2+9j}{5} \qquad \text{(2 marks)}.
\end{aligned}$$

9. $\sin^{-1}(\sqrt{3}/2) = \pi/3$ (1 mark). The general solution of $\sin\theta = \sqrt{3}/2$ is

$$\theta = (-1)^n \pi/3 + n\pi \qquad (n \in \mathbf{Z}) \qquad\qquad \text{(3 marks)}.$$

10.

$$\begin{aligned}
\mathbf{a}+\mathbf{b} &= 2\mathbf{i}+3\mathbf{k} \quad \text{(1 mark)} \\
\mathbf{a}-\mathbf{b} &= 2\mathbf{j}+3\mathbf{k} \quad \text{(1 mark)} \\
|\mathbf{a}| &= \sqrt{1^2+1^2+3^2} = \sqrt{11} \qquad \text{(1 mark)} \\
|\mathbf{b}| &= \sqrt{1^2+1^2} = \sqrt{2} \qquad \text{(1 mark)} \\
\mathbf{a}\cdot\mathbf{b} &= 1-1+0 = 0 \qquad \text{(1 mark)}.
\end{aligned}$$

Hence the angle between **a** and **b** is $\pi/2$ (1 mark).

11. The Maclaurin series expansion of $\cosh x$ is

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \qquad \text{(2 marks)}$$

Hence

(a)

$$x\cosh x = x + \frac{x^3}{2!} + \cdots \qquad \text{(2 marks)}$$

(b)

$$\cosh(2x) = 1 + 2x^2 + \frac{2x^4}{3} + \cdots \qquad \text{(2 marks)}$$

(c)

$$\cosh(x^2) = 1 + \frac{x^4}{2} + \cdots \qquad \text{(2 marks)}$$

(d)

$$(\cosh x)^2 = (1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots)^2 = 1 + x^2 + \frac{x^4}{3} + \cdots \qquad \text{(4 marks)}$$

We have

$$\cosh(0.1) = 1 + \frac{0.01}{2} + \frac{0.0001}{24} + \cdots = 1.005004$$

to 6 decimal places. (3 marks)

12. The radius of the convergence R of the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

is given by

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

provided this limit exists. In this case $a_n = (-1)^n/(n\ 5^n)$, so

$$|a_n/a_{n+1}| = \frac{(n+1)\ 5^{n+1}}{n\ 5^n} = 5\frac{n+1}{n},$$

which tends to 5 as $n \to \infty$. Hence $R = 5$. (8 marks).

When $x = 5$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n}.$$

This converges by the alternating series test, which states that

$$\sum_{n=0}^{\infty} (-1)^n a_n$$

converges if a_n is a decreasing sequence with $a_n \to 0$. (3 marks)

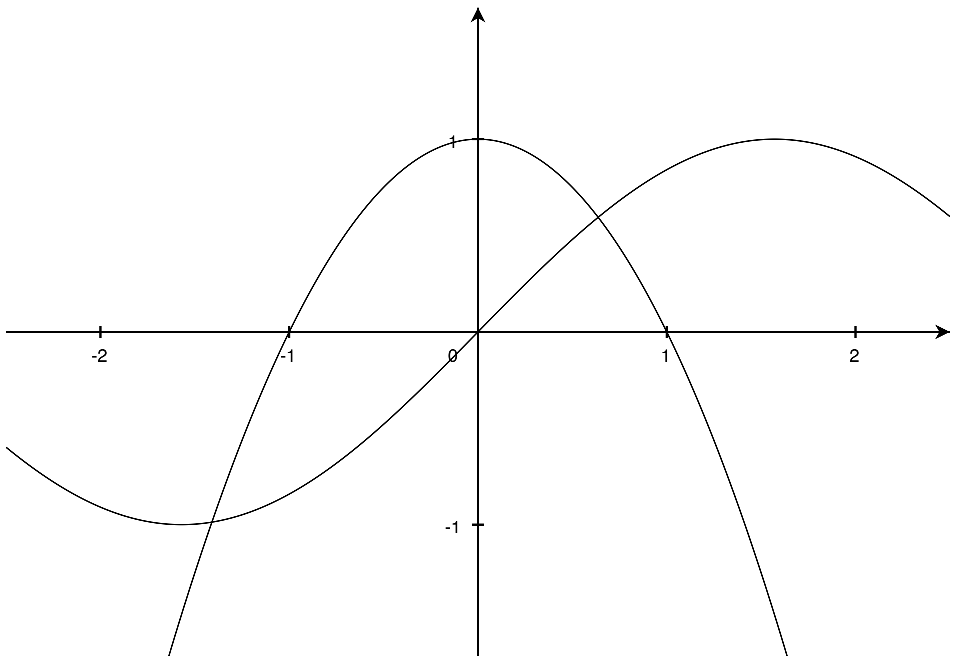
When $x = -5$, the series becomes

$$\sum_{n=0}^{\infty} \frac{1}{n},$$

which diverges (standard result). (3 marks).

Hence the series converges if and only if $-5 < x \leq 5$. (1 mark).

13. The graphs are as shown:



(6 marks).

Between 0 and 1 the function $1 - x^2$ decreases monotonically from 1 to 0 whereas $\sin x$ increases monotonically from 0 to $\sin 1 > 0$. Hence they must cross once in this range. For $x \in (1, \pi)$ there are no roots because $\sin x > 0$ and $1 - x^2 < 0$. For $x \in [\pi, \infty)$ there are no roots because $\sin x \geq -1$ and $1 - x^2 < -8$. (3 marks).

We have $f'(x) = -2x - \cos x$, so the Newton-Raphson formula becomes

$$x_{n+1} = x_n + \frac{1 - x_n^2 - \sin x_n}{2x_n + \cos x_n} \quad \text{(3 marks).}$$

Hence with $x_0 = 0.7$

$$x_1 = 0.638001.$$

$$x_2 = 0.636733.$$

$$x_3 = 0.636733.$$

(1 mark each).

14. For $x < 0$ we have $f(x) = x^2 + 3x - 1$, which has zeros at

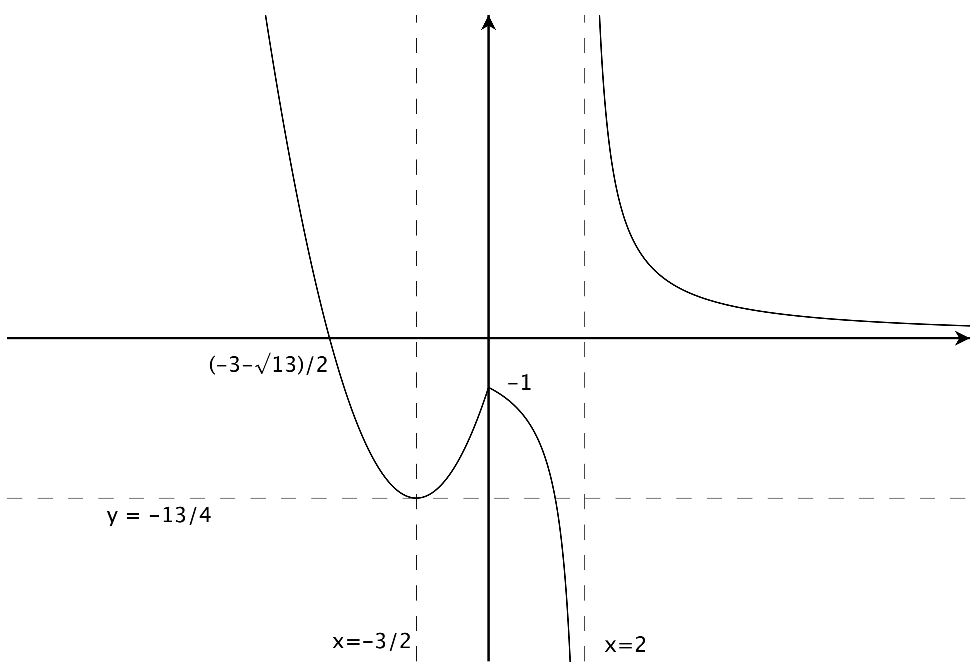
$$x = \frac{-3 \pm \sqrt{13}}{2},$$

of which $-(3 + \sqrt{13})/2$ lies in the range of definition. As $x \to 0$ we see $f(x) \to -1$.

The derivative is $f'(x) = 2x + 3$, so there is a stationary point at $x = -3/2$. Since $f''(x) = 2$, the stationary point is a local minimum. $f(x) = -13/4$ at the stationary point. The gradient of $x^2 + 3x - 1$ at $x = 0$ is 3.

For $x \geq 0$ we have $f(x) = 2/(x - 2)$, which has no zeros and is equal to -1 at $x = 0$, and tends to 0 as $x \to \infty$. $f'(x) = -2/(x - 2)^2$, so there are no stationary points, and $f(x)$ is decreasing in $(0, 2) \cup (2, \infty)$. The gradient is $-1/2$ at $x = 0$. There is a vertical asymptote at $x = 2$.

The graph of $f(x)$ is therefore



(12 marks).

$f(x)$ is not continuous at $x = 2$, since 2 is not in its maximal domain. (1 mark).

$f(x)$ is not differentiable at $x = 2$ (not in maximal domain), nor at $x = 0$ (no well-defined tangent to the graph at this point). (2 marks).

15. Let $z = \cos\theta + j\sin\theta$, so by de Moivre's theorem

$$\begin{aligned} z^n &= \cos n\theta + j\sin n\theta \\ z^{-n} &= \cos n\theta - j\sin n\theta. \end{aligned}$$

Thus $z^n + z^{-n} = 2\cos n\theta$. (4 marks)

In particular, $2\cos\theta = z + z^{-1}$ so

$$\begin{aligned} 8\cos^3\theta &= (z + z^{-1})^3 \\ &= z^3 + 3z + 3z^{-1} + z^{-3} \\ &= (z^3 + z^{-3}) + 3(z + z^{-1}) \\ &= 2\cos 3\theta + 6\cos\theta. \end{aligned}$$

Thus,

$$4\cos^3\theta = \cos 3\theta + 3\cos\theta,$$

so $a = 1$, $b = 3$ and $c = 0$. (5 marks)

So

$$\begin{aligned} \int_0^{\pi/4} \cos^3 x\, dx &= \frac{1}{4}\int_0^{\pi} (\cos(3x) + 3\cos(x))\, dx \\ &= \frac{1}{4}\left[\frac{\sin(3x)}{3} + 3\sin\, x\right]_0^{\pi/4} \\ &= \frac{1}{4}\left(\frac{\sin(3\pi/4)}{3} + 3\sin(\pi/4)\right) \\ &= \frac{1}{4}\left(\frac{\sqrt{2}}{6} + \frac{3\sqrt{2}}{2}\right) = \frac{5\sqrt{2}}{12}. \end{aligned}$$

(6 marks)