Solutions to MATH105 exam September 2011
Section A

3 marks	1.a) For a real number x, $x^2 + 2x - 3 = 0$ if and only if $x = 1$ or $x = -3$. This is true because $x^2 + 2x - 3 = (x - 1)(x + 3) = 0 \Leftrightarrow x - 1 = 0$ or $x + 3 = 0$.
3 marks	b) For a real number x, if x is greater than 2, then x is greater than 1 and less than 3. This is clearly false. For example if $x = 4$ then $4 > 2$ but it is not true that $4 < 3$.
Standard homework exercises 6 marks in total	
1 mark	2a)$\exists x \in \mathbb{R}, x \notin \mathbb{Q}$.
3 marks	b) $\exists x \in \mathbb{R}$, $x < 3 \wedge x^2 \geq 9$.
Standard homework exercises 4 marks in total	
1 mark	3a) No, $2 \notin [0, 2)$.
1 mark	b) No $3 \notin X$ because $3^2 > 5$.
1 mark	c) No $1 \notin X$.
1 mark	d) Yes
1 mark	e) No because $1 - 2i$ is not a real number
1 mark	f) No because $\sqrt{3}$ is not rational.
Standard homework exercises: no reasoning required. 6 marks in total	
1 mark	4a) $1 < 3x - 5 \Leftrightarrow 6 < 3x \Leftrightarrow 2 < x$.
2 marks	b) If $1 + x > 0$ then $-1 < \dfrac{2-x}{1+x} < 1 \leftrightarrow \dfrac{2-x}{1+x} < 1 \Leftrightarrow -1 - x < 2 - x < 1 + x \Leftrightarrow -1 < 2 < 1 + 2x \Leftrightarrow \dfrac{1}{2} < x$, which is compatible with $x > -1$.
2 marks	If $1 + x < 0$ then$-1 < \dfrac{2-x}{1+x} < 1 \leftrightarrow -1 - x > 2 - x >> 1 + x \Rightarrow -1 > 1$, which is never true. So altogether we have $-1 <\leftrightarrow \dfrac{1+x}{1-x} < 1 \Leftrightarrow \dfrac{1}{2} < x$. It is permissible to do this question by sketching the graph.
Standard homework exercises. 5 marks in total	

1 marks

5. To start the induction, $2^4 = 16 < 24 = 4!$. So $2^n < n!$ is true for $n = 4$.

Now suppose inductively that $n \geq 4$ and $2^n < n!$. Then

5 marks

$$2^{n+1} = 2 \cdot 2^n < 2 \cdot n! < (n+1) \cdot n! = (n+1)!$$

So true for n implies true for $n+1$ and $2^n < n!$ is true for all $n \geq 4$.

Standard homework exercise
6 marks in total

6. Performing integral row operations to implement Euclid's algorithm:

$$\left(\begin{array}{cc|c} 1 & 0 & 168 \\ 0 & 1 & 408 \end{array}\right) \underset{R_2 - 2R_1}{\rightarrow} \left(\begin{array}{cc|c} 1 & 0 & 168 \\ -2 & 1 & 72 \end{array}\right) \overset{R_1 - 2R_2}{\rightarrow} \left(\begin{array}{cc|c} 5 & -2 & 24 \\ -2 & 1 & 72 \end{array}\right)$$

$$\underset{R_2 - 3R_1}{\rightarrow} \left(\begin{array}{cc|c} 5 & -2 & 124 \\ -17 & 7 & 0 \end{array}\right)$$

4 marks

As a result of this:

1 mark (i) the g.c.d. d is 24;

1 mark (ii) from the last row of the last matrix, $r = 7$ and $s = 17$;

1 mark (iii) from the first row of either of the last two matrices $m = 5$ and $n = -2$;

2 marks (iv) The l.c.m is $408 \times 7 = 2856$.

Standard homework exercise
9 marks in total

3 marks

7a) $f((-1, \infty)) = (-1, \infty)$ because the cube root of x exists for all $x \in \mathbb{R}$, f is increasing, $f(-1) = -1$ and $f(x) \to +\infty$ as $x \to +\infty$. So the image of f is $(-1, , \infty)$ and f is surjective. Also, f is injective, because f is strictly increasing, and any strictly increasing function is injective.

4 marks

b) $f(x) = y \Leftrightarrow y = \dfrac{x+1}{x-1} \Leftrightarrow xy - y = x + 1 \Leftrightarrow x(y-1) = y + 1 \Leftrightarrow x = \dfrac{y+1}{y-1}$. Now $\dfrac{y+1}{y-1}$ is defined for $y \in \mathbb{R} \Leftrightarrow y \neq 1$. So the image of f is $(-\infty, 1) \cup (1, \infty) \neq \mathbb{R}$ and f is not surjective. However, f is injective, because, for any $y \neq 1$, the only value of x for which $f(x) = y$ is $x = \dfrac{y+1}{y-1}$.

Standard homework exercise
7 marks in total

3 marks	8a) One conditional definition is $\{x \in \mathbb{R} : -1 \leq x \leq 1\}$
3 marks	b) One constructive definition of this set is $\{2\sin x : x \in \mathbb{R}\}$
Standard homework exercise 6 marks in total	
1 mark	9a) This is neither increasing nor decreasing, because $x_1 = -1$, $x_2 = 2$ and $x_3 = -3$.
3 marks	b) $n^2 - 8n + 15 = (n-3)(n-5)$. So $x_1 = 8$, $x_2 = 3$ $x_3 = 0$, $x_4 = -1$ and $x_5 = 0$. So this sequence is neither increasing or decreasing.
2 marks	c) Since $\dfrac{1}{n^2}$ is decreasing with n, we see that x_n is an increasing sequence.
Standard homework exercise 6 marks in total	

Section B

Theory from lectures 4 marks	10. $\sim$ is *reflexive* if $$x \sim x \forall x \in X$$ $\sim$ is *symmetric* if $$x \sim y \ \Rightarrow \ y \sim x \ \forall \ x, \ y \in X.$$ $\sim$ is *Transitive* if $$(x \sim y \wedge y \sim z) \ \Rightarrow \ x \sim z \ \forall \ x, \ y \ \in X.$$
Standard homework exercise 4 marks	a) $\sim$ is reflexive because $3 \mid x - x = 0$ for all $x \in \mathbb{Z}$. It is symmetric because if $x - y = 3m$ for some $m \in \mathbb{Z}$ then $y - x = 3(-m)$. .It is also transitive because if $x - y = 3m$ and $y - z = 3p$ for m and $p \in \mathbb{Z}$ then $x - z = 3(m + p)$ where $m + p \in \mathbb{Z}$. So $\sim$ is an equivalence relation
Standard homework exercise 2 marks	b) If $x = 1$ then $x \in \mathbb{Z}$ and $xx = x^2 = 1$ is not even. So $\sim$ is not refelexive and hence not an equivalence relation.
Standard homework exercise 1 mark	c) If $x = \dfrac{1}{2}$ then $x \in \mathbb{Q}$ and $xx = x^2 = \dfrac{1}{4} \notin \mathbb{Z}$. So once again $\sim$ is not reflexive and not an equivalence relation.
Unseen 4 marks	d) If $z \in \mathbb{C}$ then $z - z = 0 = 0 + 0i$. So $\sim$ is reflexive. If $z - w = m + ni$ for m, $n \in \mathbb{Z}$ then $w - z = -m + (-n)i$ and $-m$, $-n \in \mathbb{Z}$ So $\sim$ is symmetric. If $z - w = m_1 + n_1 i$ and $w - v = p + qi$, where m, n, p, $q \in \mathbb{Z}$, then $z - v = (z - w) + (w - v) = m + ni + p + qi = (m + p) + (q + n)i$ and $m + p$, $n + q \in \mathbb{Z}$. So $\sim$is transitive and $\sim$ is an equivalence relation.
15 marks in total	

3 marks	11. We have $x_0 = 2$ and $x_1 = \frac{7}{4}$. So $x_1^2 - 3 = \frac{49}{16} - 3 = \frac{1}{16}$. So (i), (iii) and (iv) hold for $n = 0$ and (v) holds for $n = 1$.
2 marks	(i) If $x_n > 0$ then $\frac{x_2}{2}$ and $\frac{3}{2x_n} > 0$ and $x_{n+1} > 0$
5 marks	(ii) $$x_{n+1}^2 - 3 = \left(\frac{x_n}{2} + \frac{3}{2x_n}\right)^2 - 3 = \frac{x_n^2}{4} + \frac{3}{2} + \frac{9}{4x_n^2} - 3 = \frac{x_n^2}{4} - \frac{3}{2} + \frac{9}{4x_n^2} = \frac{1}{4x_n^2}(x_n^4 - 6x_n^2 + 9) = \frac{(x_n^2 - 3)^2}{4x_n^2}$$
1 mark	(iii) From (ii), we see that $x_{n+1}^2 - 3 > 0$
4 marks	(iv) From (iii) for $x_n$we see that $x_{n+1} - x_n = -\frac{x_n^2 - 3}{2x_n} < 0$ and hence $x_{n+1} < x_n$. From $x_{n+1} > 0$ (i) and $x_{n+1}^2 - 3 > 0$ (iii) we see that $x_{n+1} > 1$.
Standard homework exercises 15 marks in total	

Theory from lectures 3 marks	12(i) $\lvert A\cup C\cup S\rvert = \lvert A\rvert+\lvert C\rvert+\lvert S\rvert-\lvert A\cap C\rvert-\lvert A\cap S\rvert-\lvert C\cap S\rvert+\lvert A\cap C\cap S\rvert$.
Similar to homework exercises 3 marks	(ii) $A = (A \cap C) \cup (A \cap S)$ and $(A \cap C) \cap (A \cap S) = A \cap C \cap S$ Therefore, by the inclusion-exclusion principle for two sets, $$\lvert A\rvert = \lvert A \cap C\rvert + \lvert A \cap S\rvert - \lvert A \cap C \cap S\rvert$$
Unseen but similar to homework exercises 4 marks	(iii) From (i) we obtain $$30 = 24 + 26 + 26 - \lvert A \cap C\rvert - \lvert A \cap S\rvert - \lvert C \cap S\rvert + 17$$ or $$63 = \lvert A \cap C\rvert + \lvert A \cap S\rvert + \lvert C \cap S\rvert \qquad (1)$$ From (ii) we obtain $$24 = \lvert A \cap C\rvert + \lvert A \cap S\rvert - 17$$ or $$41 = \lvert A \cap C\rvert + \lvert A \cap S\rvert \qquad (2)$$ Subtracting (2) from (1) we obtain that the number of people who both canoe and sail is $\lvert C \cap S\rvert = 63 - 41 = 22$.
Unseen but similar to homework exercises 5 marks	(iv) $$\lvert S\rvert = 1 + \lvert S \cap (A \cup C)\rvert = 1 + \lvert S \cap A\rvert + \lvert S \cap C\rvert - \lvert S \cap A \cap C\rvert$$ So $$26 = 1 + \lvert S \cap A\rvert + 22 - 17$$ and $\lvert S \cap A\rvert = 20$. Then from (2) we obtain that $\lvert A \cap \lvert C\rvert = 21$
A solution obtained by drawing a Venn diagram and writing down simultaneous equations to be solved is acceptable. 15 marks in total	

Theory from lectures 6 marks	13(i) A set $A \subset Q$ is a *Dedekind cut* if • A is nonempty, and bounded above, • $x \in A \wedge y < x \Rightarrow y \in A$ • A does not have a maximal element.
Similar to homework exercises	
	medskip
1 mark 2 marks	(ii)a) $\{x \in \mathbb{Q} : x > 1\}$ is not bounded above, so not a Dedekind cut (ii)b)2 is a maximal element of $\{x \in \mathbb{Q} : x \leq 2\}$ and so A is not a Dedekind cut
Similar to practice exam	
6 marks	(iii)A is bounded above by 1, which is not in A, because $x^2+x-1 = (x+\frac{1}{2})^2 - \frac{5}{4}$ is strictly increasing for $x \geq -\frac{1}{2}$, and $1^2+1-1 > 0$. If $a \in A$ and $b < a$ then if $b < 0$ we have $b \in A$. If $0 \leq b < a$ then since x^2+x-1 is strictly increasing on $[0, \infty)$, we have $b^2+b-1 < a^2+a-1 < 0$ and $b \in A$. If $0 < a,\ \varepsilon < 1$ then $(a+\varepsilon)^2+a+\varepsilon-1 = a^2+a-1+2a\varepsilon+\varepsilon^2+\varepsilon < a^2+a-1+2\varepsilon+\varepsilon+\varepsilon$ $< a^2+a-1+4\varepsilon$ If $\varepsilon < -\dfrac{a^2+a-1}{4}$ then $a^2+a-1+4\varepsilon < 0$ and, if $\varepsilon \in \mathbb{Q}$, $a+\varepsilon \in A$. So a is not maximal, for any $a \in A$, and A is a Dedekind cut.
15 marks in total	