Notation

$\mathbb{N}$ the set of natural numbers : integers ≥ 0

$\mathbb{Z}$ set of integers, $0, \pm 1, \pm 2 \dots$

$\mathbb{Z}_+$ set of strictly positive integers, $1, 2, 3 \dots$

$\mathbb{Q}$ set of rational numbers $\frac{m}{n}$ for integers m, n with $n \neq 0$

$\mathbb{R}$ set of real numbers

$\mathbb{C}$ set of complex numbers

ϕ Empty set.

$\in$ is in e.g. $1 \in \mathbb{N}$ $-1 \in \mathbb{Z}$

$\notin$ is not in e.g. $-1 \notin \mathbb{N}$ $\neg(-1 \in \mathbb{N})$

$\{ : \}$ the set of, .. such that

If $P(x)$ is a statement about x then

$\{x : P(x)\}$ means the set of x such that $P(x)$

Often one uses $\{x \in A : P(x)\}$, meaning the set of x in A such that $P(x)$.

Examples $\{x \in \mathbb{R} : x^2 \leq 1\}$ The set of real numbers x such that $x^2 \leq 1$

$\{n \in \mathbb{Z} : n \leq -1\}$ The set of integers n such that $n \leq -1$

$\mathbb{N} = \{n \in \mathbb{Z} : n \geq 0\}$

Rather differently ~~&~~ $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$

Interval notation

If $a, b \in \mathbb{R}$, $(a, b) = \{x \in \mathbb{R} : a < x < b\}$

This is the open interval of real numbers between a and b

$(a,b) \neq \phi \Leftrightarrow a < b$

$(a,b) = \phi \Leftrightarrow a \geqslant b$

$[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ This is a closed interval - the closed interval of real numbers between a and b, including a and b.

$[a,b] \neq \phi \Leftrightarrow a \leq b$

There are also half-open intervals - which are also half-closed

$[a,b) = \{x \in \mathbb{R} : a \leq x < b\}$

$(a,b] = \{x \in \mathbb{R} : a < x \leq b\}$

There are also semi-infinite and infinite intervals

$(a,\infty) = \{x \in \mathbb{R} : a < x\}$

$[a,\infty) = \{x \in \mathbb{R} : a \leq x\}$

$(-\infty, b) = \{x \in \mathbb{R} : x < b\}$

$(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$

$(-\infty, \infty) = \mathbb{R}$

Union and intersection

$A \cup B = \{x : x \in A \vee x \in B\}$ The union of A and B

$A \cap B = \{x : x \in A \wedge x \in B\}$ The intersection of A and B.

$A \setminus B = \{x : x \in A \wedge x \notin B\}$ A minus B

There are often several different natural ways of describing sets.

Example $\{x \in \mathbb{R} : x^2 < 4\} = \{x \in \mathbb{R} : -2 < x < 2\} = (-2, 2)$

$= (-2,\infty) \cap (-\infty, 2)$

$\{x \in \mathbb{R} : x^2 \geqslant 4\} = \{x \in \mathbb{R} : x \geqslant 2 \vee x \leq -2\}$

$= (-\infty, -2] \cup [2, \infty)$

Using examples of inequalities worked out previously

$$\{x\in\mathbb{R} : x^2-3x+2 \le 0\} = \{x\in\mathbb{R}: 1\le x\le 2\} = [1,2]$$

$$\{x\in\mathbb{R}: x^2-3x+2 > 12\} = (-\infty,-2)\cup(5,\infty)$$

$$\{x\in\mathbb{R}: \left|\frac{1}{1+x}\right| < 1\} = (-\infty,-2)\cup(0,\infty)$$

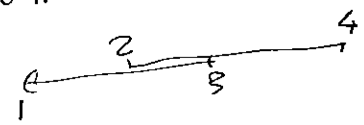
Examples of simplifying intersections and unions

$$(-\infty,2]\cap[1,\infty) = \{x\in\mathbb{R}: x\le 2\}\cap\{x\in\mathbb{R}: x\ge 1\}$$

$$= \{x\in\mathbb{R}: x\le 2 \wedge x\ge 1\} = \{x\in\mathbb{R}: 1\le x\le 2\}$$

$$= [1,2]$$

$$(1,3]\cup[2,4] = \{x\in\mathbb{R}: 1<x\le 3 \vee 2\le x\le 4\}$$



$$= (1,4]$$

$$(1,3]\cap[2,4] = [2,3]$$

$$(1,3]\setminus[2,4] = (1,2)$$

Quantifiers

Quantifiers are used for quantifying statements. The two most common ones are

$\forall$ for all / for any

$\exists$ there exists

Examples are:

$\forall x \in \mathbb{R},\ x^2 \geq 0$ For all real numbers x, $x^2 \geq 0$. (True)

$\exists x \in \mathbb{R},\ x^2 = 2$ The comma here is interpreted as "such that"
There exists a real number x such that $x^2 = 2$. There are two such real numbers, so this is true

Are the following statements true or false?

$\forall x, y \in \mathbb{R},\ x+1 < y+1 \Rightarrow x < y$
For all real numbers x and y, if $x+1 < y+1$ then $x < y$. True

$\exists x \in \mathbb{R},\ x^2 + 2 = 0$ ~~False~~ !
There exists a real number x such that $x^2+2 = 0$
False because $x^2+2 \geq 2$ for all real numbers x.

$\forall x, y \in \mathbb{R},\ x^2+1 < y^2+1 \Rightarrow x < y$
For all real numbers x and y, if $x^2+1 < y^2+1$, then $x < y$.
This is false as can be seen by working through the inequalities

$x^2+1 < y^2+1 \Leftrightarrow x^2+1-y^2-1 = x^2-y^2 < 0$

$\Leftrightarrow (x-y)(x+y) < 0 \Leftrightarrow (x-y < 0 \wedge x+y > 0) \vee (x-y > 0 \wedge x+y < 0)$

$\Leftrightarrow (x < y \wedge \underbrace{x+y > 0}_{x > -y}) \vee (x > y \wedge \underbrace{x+y < 0}_{x < -y})$

$\Leftrightarrow (-y < x < y) \vee (y < x < -y) \Leftrightarrow -|y| < x < |y|$

or simply $x^2+1 < y^2+1 \Leftrightarrow x^2 \leq y^2 \Leftrightarrow -|y| \leq x \leq |y|$

Order of quantifiers is important

$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x < y$

For any real number x, there exists a real number y such that $x < y$ (True)

$\exists x \in \mathbb{R}, x < y \; \forall y \in \mathbb{R}$

There exists a real number x such that $x < y$ for all real numbers y (False)

$\exists x \in \mathbb{N}, x < y \; \forall y \in \mathbb{N}$

There exists a natural number x such that $x < y$ for all natural numbers y (False)

$\exists x \in \mathbb{N}, x \leq y \; \forall y \in \mathbb{N}$ (True; take $x = 0$. Then $x = 0 \leq y \; \forall \; y \in \mathbb{N}$)

Negating statements involving quantifiers.

The best way to negate a statement is to write or say -or think the statement in ordinary English, then negate it and put it in logical ~~shorthand~~ language using the logical symbols.

But roughly speaking

$$\neg(\forall x, P(x)) \text{ is } \exists x, \neg P(x)$$

and $\neg(\exists x, P(x))$ is $\forall x, P(x)$

There is also a rough rule for negating statements involving $\forall$ and $\Rightarrow$

$$\neg(\forall x, P(x) \Rightarrow Q(x)) \text{ is } \exists x, P(x) \wedge \neg Q(x)$$

Examples.

$\neg(\forall x \in \mathbb{R}, \underbrace{x^2 \geq 0}_{P(x)})$ is $\exists x \in \mathbb{R}, \underbrace{x^2 < 0}_{\neg P(x)}$

$\forall x \in \mathbb{R}, x^2 \geq 0$ is <u>true</u> and $\exists x \in \mathbb{R}, x^2 < 0$ is <u>false</u>

If we just had $x^2 \geq 0$ without qualifying, then because x is not ~~data~~ given and not quantified, we could not say whether or not the statement was true.

$\neg(\exists x \in \mathbb{R}, \underbrace{x^2 + 2 = 0}_{P(x)})$ is $\forall x \in \mathbb{R}, \underbrace{x^2 + 2 \neq 0}_{\neg P(x)}$

$\forall x \in \mathbb{R}, x^2 + 2 \neq 0$ is <u>true</u> $\exists x \in \mathbb{R}, x^2 + 2 = 0$ is <u>false</u>

(If a statement is <u>true</u> its negation is <u>false</u>, and vice-versa.

$\neg(\underbrace{\forall x \in \mathbb{R}, x^2 = x \Rightarrow x = 1}_{\text{if } x \in \mathbb{R} \text{ and } x^2 = x \text{ then } x = 1})$

This is <u>false</u> because its negation is <u>true</u>

is $\underbrace{\exists x \in \mathbb{R}, x^2 = x \wedge x \neq 1}$

There exists a real number x such that ~~$x \in \mathbb{R}$ and~~ $x^2 = x$ and $x \neq 1$

This is <u>true</u> because if $x = 0$ then $0^2 = 0$ and $0 \neq 1$