Equivalence Relations

Equivalence relations are everywhere in mathematics.

An equivalence relation is denoted by $\sim$, and is defined for some pairs of elements of X.

$x \sim y$ is said as "x is equivalent to y" or "x is related to y"

Definition $\sim$ is an equivalence relation on X if

$x \sim x \ \forall x \in X$ $\quad$ $\sim$ is reflexive

$x \sim y \iff y \sim x \ \forall x, y \in X$ $\quad$ $\sim$ is symmetric

$x \sim y \wedge y \sim z \implies x \sim z \ \forall x, y, z \in X$ $\quad$ $\sim$ is transitive

Examples

① If X is any set then $=$ is an equivalence relation on X

② Let $X = \mathbb{Z}$. Define $n \sim m \iff 2 \mid n-m$. Then $\sim$ is an equivalence relation. To check this:

Reflexive $\quad n-n = 0 \quad$ So $2 \mid n-n \ \forall n \in \mathbb{Z} \quad n \sim n \ \forall n \in \mathbb{Z}$

Symmetric $\quad 2 \mid n-m \iff 2 \mid m-n \quad$ So $n \sim m \iff m \sim n \ \forall n, m \in \mathbb{Z}$

Transitive $\quad 2 \mid n-m \wedge 2 \mid m-p \implies 2 \mid (n-m)+(m-p) \implies 2 \mid n-p$

So $n \sim m \wedge m \sim p \implies n \sim p \ \forall n, m, p \in \mathbb{Z}$.

(3). Let X be a set of people.

Define $x \sim y$ if x and y have the same parents, that is, x and y are full siblings (including $x = y$)

Then $\sim$ is an equivalence relation

(4) Again, let X be a set of people.

Define $x \sim y$ if x and y have one set of grandparents in common, that is, if x and y are first cousins

Then $\sim$ is not an equivalence relation or unlikely to be, because $\sim$ is not transitive (unless X is a very small set or it is an unusual community)

(5) Non example $X = \mathbb{R}$ $\leq$ is not an equivalence relation because it is not <u>symmetric</u> e.g. $0 \leq 1$ but not $1 \leq 0$. (This relation is reflexive and transitive but since it is not symmetric it is not an equivalence relation.)

<u>Non example</u> $X = \mathbb{R}$. Define $\sim$ by $x \sim y \Leftrightarrow$ $y = x+1$.

Then $\sim$ is <u>not</u> reflexive, <u>not</u> symmetric and <u>not</u> transitive. So it is not an equivalence relation.

<u>Example</u> $X = \mathbb{R}$. Define $\sim$ by $x \sim y \Leftrightarrow |x| = |y|$

<u>Reflexive</u> $|x| = |x|$ $\forall x \in \mathbb{R}$. So $x \sim x$ $\forall x \in \mathbb{R}$

Symmetric $|x| = |y| \iff |y| = |x| \; \forall x, y \in \mathbb{R}$.

So $x \sim y \iff y \sim x \; \forall x, y \in \mathbb{R}$.

Transitive $|x| = |y| \wedge |y| = |z| \implies |x| = |z| \quad \forall x, y, z \in \mathbb{R}$.

So $x \sim y \wedge y \sim z \implies x \sim z \; \forall x, y, z \in \mathbb{R}$.

So $\sim$ is reflexive, symmetric and transitive and is an equivalence relation

Important Example $X = \mathbb{Z}$, Let $p \in \mathbb{Z}$. Define $\sim_p$ by $n \sim_p m \iff p \mid n - m$. Then $\sim_p$ is an equivalence relation.

Reflexive $n - n = 0 \quad p \mid 0$ so $n \sim_p n \; \forall n \in \mathbb{Z}$

Symmetric $p \mid n - m \iff p \mid -(n-m) \iff p \mid m - n$

So $n \sim_p m \iff m \sim_p n \quad \forall n, m \in \mathbb{Z}$

Transitive $p \mid n - m \wedge p \mid m - k \implies p \mid n - k \quad \forall n, m, k \in \mathbb{Z}$

So $n \sim_p m \wedge m \sim_p k \implies n \sim_p k \; \forall n, m, k \in \mathbb{Z}$.

Important Example $X = \mathbb{R}^2$

Define $(x_1, y_1) \sim (x_2, y_2)$ if $\exists \lambda \in \mathbb{R} \setminus \{0\}$ such that $(x_2, y_2) = \lambda(x_1, y_1)$

Reflexive $(x_1, y_1) = 1(x_1, y_1)$ So $(x_1, y_1) \sim (x_1, y_1) \; \forall (x_1, y_1) \in \mathbb{R}^2$

Symmetric For $\lambda \in \mathbb{R} \setminus \{0\}$, $(x_2, y_2) = \lambda(x_1, y_1) \iff (x_1, y_1) = \frac{1}{\lambda}(x_2, y_2)$

So $(x_1, y_1) \sim (x_2, y_2) \iff (x_2, y_2) \sim (x_1, y_1) \quad \forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$

<u>Transitive</u> If $(x_2, y_2) = \lambda(x_1, y_1)$ and $(x_3, y_3) = \mu(x_2, y_2)$ for $\lambda, \mu \in \mathbb{R}\setminus\{0\}$ then $\lambda\mu \in \mathbb{R}\setminus\{0\}$ and

$$(x_3, y_3) = \lambda\mu(x_1, y_1)$$

So $(x_1, y_1) \sim (x_2, y_2) \wedge (x_2, y_2) \sim (x_3, y_3) \implies (x_1, y_1) \sim (x_3, y_3)$
$\forall (x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2$

So $\sim$ is <u>reflexive</u>, <u>symmetric</u> and <u>transitive</u>, and is an equivalence relation.

<u>Non example</u> Define $X = \mathbb{Z}$ and $n \sim m \iff n - m$ is prime.
Then $n - n = 0$ is not prime. So $\sim$ is not reflexive and not an equivalence relation. (It is also not transitive, but is symmetric.)

<u>Equivalence Classes</u>

Let X be any set and let $\sim$ be an equivalence relation on X. Let $x \in X$.

<u>Definition</u> The <u>equivalence class of x</u> is the set $\{y \in X : y \sim x\}$
This is often denoted by $[x]$.

$$\text{So } [x] = \{y \in X : y \sim x\} = \{y \in X : x \sim y\}.$$

<u>Theorem</u> If $\sim$ is an equivalence relation on X, and $[x]$ denotes the equivalence class of x with respect to $\sim$, then

1. $\forall x, y \in X$, $[x] = [y] \iff x \sim y \iff [x] \cap [y] \neq \emptyset$
2. $X = \bigcup_{x \in X} [x]$ X is the union of equivalence classes $[x]$ for $x \in X$.

60a)

<u>Proof</u> Since $[x] = \{y : y \sim x\} = \{y : x \sim y\}$

we see that $x \sim y \Leftrightarrow y \in [x]$.

Now we show that $x \sim y \Rightarrow [x] = [y]$.

First we show that $x \sim y \Rightarrow [y] \subset [x]$.

So suppose $x \sim y$ and $z \in [y]$.

Then $x \sim y$ and $y \sim z$. So $x \sim z$ (transitivity)

So $z \in [x]$.

So $z \in [y] \Rightarrow z \in [x]$.

That is $[y] \subset [x]$.

Similarly $x \sim y \Rightarrow y \sim x \Rightarrow [x] \subset [y]$.

So $x \sim y \Rightarrow [x] \subset [y] \wedge [y] \subset [x] \Rightarrow [x] = [y]$.

Clearly $[x] = [y] \Rightarrow x \sim y$ because $x \in [x]$ and

so $[x] = [y] \Rightarrow x \in [y] \Rightarrow x \sim y$.

So $[x] = [y] \Leftrightarrow x \sim y$.

Also $[x] = [y] \Rightarrow [x] \cap [y] \neq \phi$

But $[x] \cap [y] \neq \phi \Rightarrow \exists z,\ z \in [x] \wedge z \in [y]$

$\Rightarrow \exists z,\ x \sim z \wedge z \sim y \Rightarrow x \sim y$

So $[x] \cap [y] \neq \phi \Rightarrow x \sim y$

and $x \sim y \Leftrightarrow [x] = [y] \Leftrightarrow [x] \cap [y] \neq \phi$

2. $x \in [x]\ \forall x \in X$.

So $X = \bigcup_{x \in X} [x]$.

<u>Examples of equivalence classes</u>

1. $X = \mathbb{Z}$ and define $n \sim m \iff 2 \mid n-m$

This is an equivalence relation, as shown earlier

All even numbers are equivalent. All odd numbers are equivalent

$[0] = \{n \in \mathbb{Z} : 2 \mid n\} = [2] = [n]$ for any even integer n.

set of even integers.

$[1] = \{n \in \mathbb{Z} : 2 \mid n-1\} = \{2k+1 : k \in \mathbb{Z}\} = [n]$ for

any odd integer n.

So there are exactly 2 equivalence classes, the set of even integers and the set of odd integers.

2. If $X = \mathbb{Z}$ and $p \in \mathbb{Z}_+$ and $n \sim_p m \iff p \mid n-m$

then $\sim_p$ is an equivalence relation and

$[n] = \{m \in \mathbb{Z} : m \sim_p n\} = \{n + kp : k \in \mathbb{Z}\}$

There are exactly p equivalence classes

$\{kp + r : k \in \mathbb{Z}\} = [r]$ for $0 \leq r < p$.

In Humphreys + Prest, the equivalence class of r is denoted by $[r]_p$

3. If $X = \mathbb{R}^2$ and $(x_1, y_1) \sim (x_2, y_2) \iff (x_2, y_2) = \lambda(x_1, y_1)$ for $\lambda \in \mathbb{R} \setminus \{0\}$

then the equivalence classes are $\{(0,0)\}$ and the lines through $(0,0)$ with $(0,0)$ removed.