Solutions to MATH105 Practice Exam
Section A

Marks	Solution
2 marks	1a) (ii) and (iv) are logically equivalent to (i). (iii) is not, because $\vee$ means "or". (v) is not, because $x^2 \leq 16 \Leftrightarrow x \in [-4, 4]$.
2 marks	b) (iii), (iv) and (v) are logically equivalent to (i). (ii) is not, but the statement $x \in A \Rightarrow x \in B$ is logically equivalent to (i).
2 marks	c)(iii) and (iv) are logically equivalent to (i). (v) is not, because, for example, (v) holds when $m = 4$ and $n = 2$ but (i) is not true for this choice of m and n, because 4 does not divide 2.
6 marks in total	
3 marks	2a) For a real number x, if $x < 5$ then $x^2 < 25$. This is false, because $-6 < 5$ but $(-6)^2 = 36 > 25$.
3 marks	b) For a real number x, if x is greater than 0 or less than -1, then x is greater than 0. This is clearly false because if $x < -1$ then it is true that "$x < -1$ or $x > 0$". But it is not true that $x > 0$.
6 marks in total	
1 mark	3a)$x \leq 1 \vee x \geq 2$.
3 marks	b) $\exists x \in \mathbb{R},\ x^2 \leq -2$. [Of course, this is false, but that was not what was asked.]
4 marks in total	
1 mark	4a) $2 + 3x < -1 \Leftrightarrow 3x < -3 \Leftrightarrow x < -1$.
2 marks	b) If $3 - x > 0$ then $1 <\leftrightarrow \dfrac{2+x}{3-x} < 2 \Leftrightarrow 3 - x < 2 + x < 6 - 2x \Leftrightarrow$ $(1 < 2x \wedge 3x < 4 \Leftrightarrow \dfrac{1}{2} < x < \dfrac{4}{3}$, which is compatible with $3 - x > 0$.
2 marks	If $3 - x < 0$ then $1 <\leftrightarrow \dfrac{2+x}{3-x} < 2 \Leftrightarrow 3 - x > 2 + x > 6 - 2x \Leftrightarrow$ $1 > 2x \wedge 3x > 4 \Leftrightarrow \dfrac{1}{2} > x \wedge x > \dfrac{4}{3}$. This is never true. So altogether we have $1 <\Leftrightarrow \dfrac{2+x}{3-x} < 2 \Leftrightarrow \dfrac{1}{2} < x < \dfrac{4}{3}$
5 marks in total	

1 marks — 5. To start the induction, $5^2 + 1 = 26 < 32 = 2^5$. So $n^2 + 1 < 2^n$ is true for $n = 5$.

Now suppose inductively that $n \geq 5$ and $n^2 + 1 < 2^n$. Then

5 marks

$$(n+1)^2 + 1 = n^2 + 2n + 2 < 2n^2 + 2 < 2 \cdot 2^n = 2^{n+1}$$

So true for n implies true for $n+1$ and $n + 1 < 2^n$ is true for all $n \geq 5$.

6 marks in total

6.

$$\begin{array}{cc|c} 1 & 0 & 330 \\ 0 & 1 & 225 \end{array} \quad \overset{R_1 - R_2}{\rightarrow} \quad \begin{array}{cc|c} 1 & -1 & 105 \\ 0 & 1 & 225 \end{array} \quad \underset{R_2 - 2R_1}{\rightarrow} \quad \begin{array}{cc|c} 1 & -1 & 105 \\ -2 & 3 & 15 \end{array}$$

$$\overset{R_1 - 7R_2}{\rightarrow} \quad \begin{array}{cc|c} 15 & -22 & 0 \\ -2 & 3 & 15 \end{array}$$

4 marks

As a result of this:

1 mark — (i) the g.c.d. d is 15;

1 mark — (ii) from the first row of the last matrix, $r = 22$ and $s = 15$;

1 mark — (iii) from the second row of either of the last two matrices, $m = -2$ and $n = 3$;

2 marks — (iv) The lcm is $330 \times 15 = 4950$.

9 marks in total

3 marks — 7a) $f(\mathbb{R}) = \mathbb{R}$ because if x is the real cube root of $y - 1$ then $x^3 + 1 = y$. So f is surjective. Also f is injective because f is strictly increasing.

4 marks — b) $f(x) = y \Leftrightarrow y = \dfrac{2-x}{x+1} \Leftrightarrow 2 - x = xy + y \Leftrightarrow x(y+1) = 2 - y \Leftrightarrow x = \dfrac{2-y}{y+1}$ – which is, in fact, $f(y)$. Now $\dfrac{2-y}{y+1}$ is defined for $y \in \mathbb{R}$ $\Leftrightarrow y \neq -1$. So the image of f is $(-\infty, -1) \cup (-1, \infty) \neq \mathbb{R}$ and f is not surjective. However, f is injective, because, for any $y \neq -1$, the only value of x for which $f(x) = y$ is $x = \dfrac{2-y}{y+1}$.

Note that in this example, the function f is its own inverse, but this does not always happen.

7 marks in total

3 marks	8a) Since the image of the map $f(x) = e^x + 1$ is the set $(1, \infty)$, a conditional definition of this set is $\{x \in \mathbb{R} : x > 1\}$.
3 marks	b) For $n \in \mathbb{Z}$ $$2 \vert n \wedge 3 \vert n \Leftrightarrow 6 \vert n \Leftrightarrow \exists k \in Z \text{ such that } n = 6k.$$ So a constructive definition of this set is $\{6k : k \in \mathbb{Z}\}$.
6 marks in total	
1 mark	9a)This is an increasing sequence since $3(n+1)+2 = 3n+5 > 3n+2$ for all integers $n \geq 1$.
3 marks	b) $x_n = n^2 - 3n - 4 = (n+1)(n-4)$. So $n + 1 > 0$ for all $n \geq 1$ and both $n + 1$ and $n - 4$ are increasing with n. It follows that x_n is increasing.
2 marks	c) $x_n = \dfrac{(-)^n}{n^2 + 1}$ is neither increasing nor decreasing, since the terms are alternatively strictly positive and strictly negative.
6 marks in total	

Section B

4 marks	10. $\sim$ is *reflexive* if $$x \sim x \forall x \in X$$ $\sim$ is *symmetric* if $$x \sim y \;\Rightarrow\; y \sim x \;\forall\; x,\; y \in X.$$ $\sim$ is *transitive* if $$(x \sim y \wedge y \sim z) \;\Rightarrow\; x \sim z \;\forall\; x,\; y,\; \in X.$$
3 marks	a) $\sim$ is reflexive because $x-x=0 \in \mathbb{Z}$ for all $x \in \mathbb{Q}$. It is symmetric because if $x-y \in \mathbb{Z}$ then $y-x = -(x-y) \in \mathbb{Z}$. It is transitive because if $x-y \in \mathbb{Z}$ and $y-z \in \mathbb{Z}$ then $x-z = (x-y)+(y-z) \in \mathbb{Z}$. So $\sim$ is an equivalence relation.
1 mark	b) $\sim$ is not reflexive because, for example, if $x = \frac{1}{2} \in \mathbb{Q}$, then $x - 2x = -\frac{1}{2} \notin \mathbb{Z}$. So $\sim$ is not an equivalence relation
4 marks	c) If f is any polynomial with real coefficients, $f \sim f$ because $f - f = 0$ is a real constant. So $\sim$ is reflexive. If f and g are any polynomials with real coefficients, $f \sim g \Leftrightarrow f - g = c \in \mathbb{R} \Leftrightarrow g - f = -c \in \mathbb{R} \Rightarrow g \sim f$. So $\sim$ is symmetric. If f, g and h are any polynomials with real coefficients, and $f \sim g$ and $g \sim h$, then $f - g = c_1 \in \mathbb{R}$ and $g - h = c_2 \in R$ and hence $f - h = c_1 + c_2 \in \mathbb{R}$ and $f \sim h$ So $\sim$ is transitive Hence $\sim$ is an equivalence relation.
3 marks	The equivalence class of f_1 is all polynomials $x + c$, for $c \in \mathbb{R}$. The equivalence class of f_2 is all constant polynomials, that is, all polynomials of the form c, for $c \in \mathbb{R}$.
15 marks in total	

1 mark

11a) **Base Case** $x_0 = 1 = 2 \cdot 3^0 - 1$, so the formula holds for $n = 0$.

4 marks

Inductive Step Assume that $x_n = 2 \cdot 3^n - 1$. Then, using the definition for the first equality,

$$x_{n+1} = 3x_n + 2 = 3(2 \cdot 3^n - 1) + 2 = 2 \cdot 3^{n+1} - 3 + 2 = 2 \cdot 3^{n+1} - 1.$$

So

$$x_n = 2 \cdot 3^n - 1 \quad \Rightarrow \quad x_{n+1} = 2 \cdot 3^{n+1} - 1.$$

1 mark

So by induction $x_n = 2 \cdot 3^n - 1$ holds for all $n \in \mathbb{N}$.

1 mark

b) **Base Case** If $n = 1$, any function $f : \{1\} \to \mathbb{R}$ attains its maximum at 1, because, trivially, $f(i) \leq f(1)$ for all i with $1 \leq i \leq 1$ (that is, for $i = 1$).

5 marks

Inductive Step Let $n \in \mathbb{Z}_+$, and assume that any real-valued function with domain $\{i \in \mathbb{Z}_+ : 1 \leq i \leq n\}$ attains a maximum value. Now let $f : \{i \in \mathbb{Z}_+ : 1 \leq i \leq n+1\} \to \mathbb{R}$ be any function. Then the restriction of this function to $\{i \in \mathbb{Z}+_{:} 1 \leq i \leq n\}$ does attain its maximum, that is, there is $k_1 \in \mathbb{Z}_+$ with $1 \leq k_1 \leq n$ such that $f(i) \leq f(k_1)$ for all $1 \leq i \leq n$. Now define $k = k_1$ if $f(n+1) \leq f(k_1)$ and $k - n + 1$ if $f(k_1) < f(n+1)$. Then $f(i) \leq f(k)$ for all $1 \leq i \leq n+1$, that is, f attains its maximum. So

$$\text{True for } n \quad \Rightarrow \quad \text{True for } n+1.$$

1 mark

So by induction, for any $n \in \mathbb{Z}_+$, any function $f : \{k \in \mathbb{Z}_+ : 1 \leq k \leq n\} \to \mathbb{R}$ attains its maximum.

2 marks

An example of a function $f : \mathbb{Z}_+ \to [0, 1]$ which does not attain its maximum is the function f defined by

$$f(n) = 1 - \frac{1}{n}$$

for all $n \in \mathbb{Z}_+$, because $\lim_{n \to \infty} f(n) = 1$, but $1 \neq f(k)$ for any $k \in \mathbb{Z}_+$.

15 marks in total

3 marks

12(i) $|E \cup T \cup M| = |E| + |T| + |M| - |E \cap T| - |T \cap M| - |E \cap M| + |E \cap T \cap M|$.

4 marks

(ii)The number of people going on at least two tours is

$$|(E \cap T) \cup (T \cap M) \cup (E \cap M)|.$$

The intersection of any two of the sets $E \cap T$, $T \cap M$, $E \cap M$ is $E \cap T \cap M$. So the intersection of all three of these sets is also $E \cap T \cap M$. Applying the inclusion-exclusion principle we have

$$|(E \cap T) \cup (T \cap M) \cup (E \cap M)|$$

$$= |E \cap T| + |T \cap M| + |E \cap M| - 3|E \cap T \cap M| + |E \cap T \cap M|$$
$$= |E \cap T| + |T \cap M| + |E \cap M| - 2|E \cap T \cap M|.$$

4 marks

(iii) Adding the equations from (i) and (ii) the terms $|E \cap T| + |T \cap M| + |E \cap M|$ cancel and we obtain

$$28 + 18 = 46 = 22 + 7 + 21 - |E \cap M \cap T| = 50 - |E \cap M \cap T|$$

So the number $|E \cap M \cap T|$ of people going on all three tours is 4.

4 marks

(iv)$|E \cap (T \cup M)| = 22 - 6 = 16$. Applying the inclusion-exclusion principle to the two sets $E \cap T$ and $E \cap M$ we have

$$16 = |E \cap M| + |E \cap T| - |E \cap M \cap T| = 16 + |E \cap T| - 4.$$

So the number of people going on both the London Eye and Tower of London tours is

$$|E \cap T| = 16 - 12 = 4.$$

15 marks in total

6 marks

13(i) A set $A \subset Q$ is a *Dedekind cut* if

- A is nonempty, and bounded above,
- $x \in A \wedge y < x \Rightarrow y \in A$
- A does not have a maximal element.

(ii)

1 mark

(ii)a) Q is not bounded above, so not a Dedekind cut

2 marks

(ii)b) $\frac{1}{2} \in \mathbb{Q}$ but $0 \notin A$ and $0 < \frac{1}{2}$, so A is not a Dedekind cut

6 marks

(iii) A is bounded above – by 3 for example because if $a \geq 3$ then $a^2 \geq 9 > 5$. and since $x \mapsto x^2$ is strictly increasing for $x \geq 0$, if $a \in A$ and $b < a$ then either $b < 0$ – in which case $b \in A$ — or $0 \leq b^2 < a^2$ and so $a \in A$.
Also, $2 \in A$, because $2^2 = 4 < 5$.
Finally, A has no maximal element. For suppose $a \in A$ and $a \geq 2$. If $0 < \varepsilon < 1$ then $(a+\varepsilon)^2 = a^2 + 2a\varepsilon + \varepsilon^2 < a^2 + 3a\varepsilon$. If in addition $\varepsilon < \frac{5-a^2}{3a}$ then $3a\varepsilon \leq 5 - a^2$ and hence $(a+\varepsilon)^2 < 5$. If in addition $\varepsilon \in \mathbb{Q}$, then $a + \varepsilon \in \mathbb{Q}$ and $a + \varepsilon \in A$. So a is not maximal in A for any $a \in A$, and A does not have a maximal element. So A satisifies all the conditions of (i), and A is a Dedekind cut.

15 marks in total

1 mark

14. A is *finite* if either A is empty or there is $n \in \mathbb{Z}_+$ and a bijection $f : \{k \in \mathbb{Z}_+\} \to A$.

2 marks

A is *countable* if either A is finite or there is a bijection $f : \mathbb{N} \to A$ (or a bijection from $\mathbb{Z}_+$ to A)

2 marks

A and B has the same cardinality if there is a bijection $f : A \to B$.

2 marks

$\mathbb{R}$ is uncountable and $\mathbb{Z}$ and $\mathbb{Q}$ are countable.

3 marks

Schröder-Bernstein Theorem: If A and B are two sets and there are injective maps $f : A \to B$ and $g : B \to A$ then there is a bijection $h : A \to B$.

2 marks

If f is given by $f(x,0) = e^x$ and $f(0,y) = -e^y$ for $y \neq 0$ then the set of values of $f(x,0)$ is $(0,\infty)$ and the set of values of $f(0,y)$ for $y \neq 0$ is $(-\infty,-1) \cup (-1,0)$, because e^x is increasing and $-e^y$ is decreasing and $-e^0 = -1$. So altogether the image of f is $(-\infty,-1) \cup (-1,0) \cup (0,\infty)$.

3 marks

Since f is injective restricted to $\mathbb{R} \times \{0\}$ and $\{0\} \times (\mathbb{R} \setminus \{0\})$, and the images of these two sets are disjoint, the map f is injective on X. Also $g : \mathbb{R} \to X$ defined by $g(x) = (x,0)$ is injective. So there is a bijection $h : X \to \mathbb{R}$ by the Schröder-Bernstien Theorem.

15 marks in total