UNIVERSITY OF LIVERPOOL

MATH 105 Practice exam 2014

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TIME ALLOWED: The real exam will be two and a half hours in length.

This practice exam is provided so that you can familiarise yourselves with the format of the real exam. In the real exam, full marks will be obtained by complete answers to all questions in Section A and three questions in Section B. The best 3 answers in Section B will be taken into account. However, note that on the real exam there are only 4 questions on Section B. This practice exam has five.

Although this exam has the same format as the real exam (apart from the extra question) this is not the real exam! The real exam will have different questions. When you have the real exam, be careful to read the questions in front of you, NOT any questions that you may have practised on.

SECTION A

1. In each of a)-c), identify which of the statements (ii)-(v) are logically equivalent to (i). There may be more than one, or none at all. You are not required to give any reasoning, nor to determine whether any of the statements are true. In a), x is a real number. In b), A and B are subsets. In c), m and n are integers.

a) (i) $x \in [0, 4]$.

(ii) $0 \leq x \leq 4$.

(iii) $x \geq 0 \vee x \leq 4$.

(iv) $x \geq 0 \wedge x \leq 4$.

(v) $x^2 \leq 16$.

b) (i) $A \subset B$.

(ii) $x \in B \Rightarrow x \in A$.

(iii) $x \notin B \Rightarrow x \notin A$.

(iv) $A \setminus B = \emptyset$.

(v) $A \cap B = B$.

c) (i) $m \mid n$.

(ii) $m = kn$ for some $k \in \mathbb{Z}$.

(iii) $n = km$ for some $k \in \mathbb{Z}$.

(iv) $(k \in \mathbb{Z} \wedge k \mid m) \Rightarrow k \mid n$.

(v) If k is a prime and $k \mid m$, then $k \mid n$.

[6 marks]

2. Write down each of the following statements in ordinary English, and determine whether each one is true.

a) For $x \in \mathbb{R}$, $x < 5 \Rightarrow x^2 < 25$

b) For $x \in \mathbb{R}$, $(x > 0 \vee x < -1) \Rightarrow x > 0$

[6 marks]

3. Negate each of the following statements, using logical symbols where possible.

a) $x > 1 \wedge x < 2$.

b) $\forall x \in \mathbb{R},\ x^2 > -2$.

[4 marks]

4. In each of the following, find the set of all $x \in \mathbb{R}$ satisfying the inequalities.

a) $2 + 3x < -1$

b) $1 < \dfrac{2+x}{3-x} < 2$, $x \neq 3$.

[5 marks]

5. Show by induction that $n^2 + 1 < 2^n$ for all integers $n \geq 5$.

[6 marks]

6. Let $a = 330$ and $b = 225$. Using the Euclidean algorithm or otherwise, find:

(i) the g.c.d. d of a and b;

(ii) integers r and s such that $a = dr$ and $b = ds$;

(iii) integers m and n such that $d = ma + nb$;

(iv) the l.c.m. of a and b

[9 marks]

7. Determine the images of the following functions, and also determine whether the functions are injective, surjective or neither.

a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 + 1$.

b) $f : (-\infty, -1) \cup (-1, \infty) \to \mathbb{R}$ given by $f(x) = \dfrac{2-x}{x+1}$.

[7 marks]

8.

a) Find a conditional definition of the set $\{e^x + 1 : x \in \mathbb{R}\}$.

b) Find a constructive definition for the set $\{n \in \mathbb{Z}_+ : 2|n \wedge 3|n\}$.

[6 marks]

9. Determine which of the following sequences are increasing, which are decreasing, and which are neither.

a) $x_n = 3n + 2$, $n \geq 1$.

b) $x_n = n^2 - 3n - 4$, $n \geq 1$.

c) $x_n = \dfrac{(-1)^n}{n^2 + 1}$, $n \geq 1$.

[6 marks]

SECTION B

10. An equivalence relation $\sim$ on X – where $x \sim y$ means "x is equivalent to y" is *reflexive*, *symmetric* and *transitive*. Define what each of these three terms means.

In each a) and b), determine whether $\sim$ is an equivalence relation on X.

a) $X = \mathbb{Q}$ and $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$.

b) $X = \mathbb{Q}$ and $x \sim y \Leftrightarrow x - 2y \in \mathbb{Z}$.

c) Let X is the set of all polynomials f with real coefficients, and for f, $g \in X$, define $f \sim g$ to mean that $f - g$ is a real constant. Show that $\sim$ is an equivalence relation. Determine the equivalence classes of the polynomials f_1 and f_2, where $f_1(x) = x$ for all x and $f_2(x) = 0$ for all x.

[15 marks]

11.

a) Let the sequence x_n defined inductively, for all $n \in \mathbb{N}$, by $x_0 = 1$ and $x_{n+1} = 3x_n + 2$. Prove by induction on n that

$$x_n = 2 \cdot 3^n - 1 \;\; \forall \;\; n \in \mathbb{N}.$$

b) Prove by induction that, for all $n \in \mathbb{Z}_+$, a function

$$f : \{i \in \mathbb{Z}_+ : 1 \le i \le n\} \to \mathbb{R}$$

attains its maximum, that is, there is $k \in \mathbb{Z}_+$ with $1 \le k \le n$ such that

$$f(i) \le f(k) \;\; \forall \;\; 1 \le i \le n.$$

Give an example of a function $f : \mathbb{Z}_+ \to [0, 1]$ which does not attain its maximum.

12. On a week's package holiday in London, three optional tours are available: to the London Eye, the Tower of London and Madame Tussaud's. 28 people go on at least one tour and 18 go on at least two. 22 people go on the London Eye, 7 go to the Tower of London and 21 go to Madame Tussaud's. All of these might also go on other tours. Also, 6 people only go on the London Eye, and 16 go on both the London Eye and Madame Tussaud tours, some of whom also go to the Tower of London.

Let E, T and M be the sets of people going to the London Eye, the Tower of London and Madame Tussaud's respectively

(i) State the inclusion-exclusion principle for three sets. You may call these sets E, T and M, and it might be convenient to do so.

(ii) Show that the number of people going on at least two tours is

$$|E \cap T| + |T \cap M| + |E \cap M| - 2|E \cap T \cap M|$$

Hint: Apply the inclusion-exclusion principle to the three sets $E \cap T$, $T \cap M$ and $E \cap M$.

iii) Find the number of people going on all three tours.

Hint: Add equations from (i) and (ii)

(iv) Find the number of people having going on both the London Eye and Tower of London tours.

Hint: Apply the inclusion exclusion principle to the sets $E \cap T$, $E \cap M$.

13.

(i) Give the definition of a *Dedekind cut.*

(ii) Determine which (if any) of the following sets A are Dedekind cuts. Give brief reasons for your answers.

a) $A = \{x \in \mathbb{Q} : x > 0\}$

b) $A = \{x \in \mathbb{Q} : 0 < x < 1\}$

(iii) Show that $A = \{x \in \mathbb{Q} : x < 0 \vee x^2 < 5\}$ has no maximal element, and hence, or otherwise, show that A is a Dedekind cut.

Hint: If $2 \le a \in A$, and $0 < \varepsilon < 1$ show that $(a+\varepsilon)^2 < a^2 + 3a\varepsilon$. Hence show that if, in addition, $\varepsilon \in \mathbb{Q}$ and $\varepsilon \le \dfrac{5-a^2}{3a}$, then $a + \varepsilon \in A$.

14.

Define what it means for a set A to be *finite*, and what it means for A to be *countable*, and what it means for two sets to *have the same cardinality*

State which of $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Q}$ are countable and which are uncountable. No proofs are required.

State the Schröder-Bernstein theorem. This may be useful in the following.

Let $X = (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R})$. Consider the function $f : X \to \mathbb{R}$ defined by

$$f(x, 0) = e^x,$$

$$f(0, y) = -e^y \text{ if } y \neq 0.$$

Determine the image of f. Show that X and $\mathbb{R}$ have the same cardinality.