Solutions to MATH105 exam September 2012
Section A

2 marks 4 marks Standard homework exercises 6 marks in total	1.a) If $x^2 = 4$ then x is an integer. This is true because $x^2 = 4 \Leftrightarrow x = \pm 2$ and both 2 and -2 are integers. b) If x is rational and $x < 1$ then there exists a rational number y which is strictly between x and 1 This is also true, because $y = \dfrac{1+x}{2}$ is rational and $y - x = \dfrac{1-x}{2} > 0$ and $1 - y = \dfrac{1-x}{2} > 0$.
2 marks 2 marks Standard homework exercises 4 marks in total	2a)$x \leq -1 \vee x \geq 0$ b) $\exists x \in (0, \pi/2)$ such that $\tan x \leq x$.
2 marks 2 marks 2 marks Standard homework exercises 6 marks in total	3a) $-1 < 0$ and $2 < 3 < 4$. So $((-3,-1) \cup (2,4]) \cap [0,3] = (2,3]$. b) $(0,2) \cup ((1,3) \cap [2,4)) = (0,2) \cup [2,3) = (0,3)$. c) $[2,3) \subset [-1,3]$, and so $[-1,3] \cup [2,3) = [-1,3]$. $$([-1,3] \cup [2,3) \cup (6,7]) \cap [3,5) = ([-1,3] \cup (6,7]) \cap [3,5)$$ $$= ((-1,3] \cap [3,5)) \cup ((-1,3) \cap (6,7])$$ $$= (-1,3] \cap [3,5) = \{3\}.$$
4 marks 4 marks Standard homework exercises. 8 marks in total	4a) $3x^2 > 2x + 1 \ \Leftrightarrow 3x^2 - 2x - 1 > 0 \Leftrightarrow (3x+1)(x-1) > 0$ $\Leftrightarrow (3x+1 > 0 \wedge x - 1 > 0) \vee (3x+1 < 0 \wedge x - 1 < 0) \Leftrightarrow x > 1 \vee x < -\dfrac{1}{3}$. b) $\left|2 + \dfrac{3}{x}\right| \leq 1 \Leftrightarrow -1 \leq 2 + \dfrac{3}{x} \leq 1 \Leftrightarrow -3 \leq \dfrac{3}{x} \leq -1$ $$\Leftrightarrow x < 0 \wedge -3x \geq 3 \wedge 3 \geq -x \Leftrightarrow -3 \leq x \leq -1.$$

1 marks	5. To start the induction, $2^3 = 8 < 16 = 4^2$ So $n^3 < 4^n$ holds for $n = 2$. Now suppose inductively that $n^3 < 4^n$ for some $n \in \mathbb{N}$ with $n \geq 2$. Then
4 marks	$$(n+1)^3 = n^3\left(1+\frac{1}{n}\right)^3 \leq n^3\left(\frac{3}{2}\right)^3 = \frac{27}{8}n^3 < 4n^3 < 4 \times 4^n = 4^{n+1}$$ So if $n \in \mathbb{N}$ and $n \geq 2$ then $n^3 < 4^n \Rightarrow (n+1)^3 < 4^{n+1}$. So by induction $n^3 < 4^n$ for all $n \in \mathbb{N}$ with $n \geq 2$.
1 mark	For $n = 0$ we have $0^3 = 0 < 1 = 4^0$ and for $n = 1$ we have $1^3 = 1 < 4^1 = 4$.
Standard homework exercise 6 marks in total	
	6. $$\left(\begin{array}{cc\|c} 1 & 0 & 567 \\ 0 & 1 & 387 \end{array}\right) \xrightarrow{R_1 - R_2} \left(\begin{array}{cc\|c} 1 & -1 & 180 \\ 0 & 1 & 387 \end{array}\right) \xrightarrow[R_2 - 2R_1]{} \left(\begin{array}{cc\|c} 1 & -1 & 180 \\ -2 & 3 & 27 \end{array}\right)$$ $$\xrightarrow{R_1 - 6R_2} \left(\begin{array}{cc\|c} 13 & -19 & 18 \\ -2 & 3 & 27 \end{array}\right) \xrightarrow[R_2 - R_1]{} \left(\begin{array}{cc\|c} 13 & -19 & 18 \\ -15 & 22 & 9 \end{array}\right)$$ $$\xrightarrow{R_1 - 2R_2} \left(\begin{array}{cc\|c} 43 & -63 & 0 \\ -15 & 22 & 9 \end{array}\right)$$
4 marks	As a result of this:
1 mark	(i) the g.c.d. d is 9;
1 mark	(ii) from the first row of the last matrix, $r = 63$ and $s = 43$;
1 mark	(iii) from the second row of either of the last two matrices $m = -15$ and $n = 22$;
2 marks	(iv) The lcm is $567 \times 43 = 24381$.
Standard homework exercise 9 marks in total	

2 marks 3 marks 3 marks 2 marks Standard theory followed by standard homework exercises 10 marks in total	7 $f : X \to Y$ is injective if $\forall x_1,\ x_2 \in X,\ f(x_1) = f(x_2 \Leftrightarrow x_1 = x_2$. The image of f, $\mathrm{Im}(f)$ is $\{f(x) : x \in X\}$. f is a *bijection* if f is injective and $\mathrm{Im}(f) = Y$, that is, f is also surjective a) Since f is strictly decreasing on $[0, \infty)$, it is injective. We have $f(0) = 1$ and $\lim_{x\to\infty} \frac{1}{1+x^2} = 0$ So $\mathrm{Im}(f) = (0, 1]$. b) $2\cos(x) = 2\cos(-x)$ for all $x \in \mathbb{R}$. So f is not injective and $f([-\pi/2, \pi/2]) = f([0, \pi/2])$. On $[0, \pi/2]$, f is strictly decreasing, with $f(0) = 2$ and $f(\pi/2) = 0$. So $\mathrm{Im}(f) = [0, 2]$.
Standard theory 2 marks Nothing set quite like this 4 marks 6 marks in total	8. $\|A_1 \cup A_2\| = \|A_1\| + \|A_2\| - \|A_1 \cap A_2\|$ (i) Let A_1 denote the set of companies offering holidays in New York and A_2 the set of companies offering holidays in Florida. Then $15 = \|A_1\| + \|A_2\| - 8$, and $\|A_1\| = \|A_2\| + 3$. Then $2\|A_2\| = 20$ and $\|A_2\| = 10$ and $\|A_1\| = 13$.

Section B

Theory from lectures 3 marks	9. $\sim$ is *reflexive* if $$x \sim x \forall x \in X$$ $\sim$ is *symmetric* if $$x \sim y \;\Rightarrow\; y \sim x \;\forall\; x,\; y \in X.$$ $\sim$ is *transitive* if $$(x \sim y \wedge y \sim z) \;\Rightarrow\; x \sim z \;\forall\; x,\; y,\; z \in X.$$
Standard homework exercise 2 marks	(i) If $x = 1$ and $y = 0$ the x, $y \in \mathbb{R}$ and $x - y = 1 > 0$ and so $x \sim y$, but $y - x < 0$ so it is not true that $y \sim x$ and so $\sim$is not an equivalence relation.
Standard homework exercise 4 marks	(ii)For any $x \in X$ we have $x/x = 1 > 0$, so $x \sim x$. Therefore $\sim$ is reflexive. If $x \sim y$ then $x/y > 0$ and hence $y/x = (x/y)^{-1} > 0$. So $x \sim y \Rightarrow y \sim x$ and $\sim$ is symmetric. If $x \sim y$ and $y \sim z$ then $x/y > 0$ and $y/z > 0$, and hence $x/z = (x/y) \cdot (y/z) > 0$ and $x \sim z$. So $\sim$ is transitive and is an equivalence relation.
2 marks	There are two equivalence classes: $(0, \infty)$ and $(-\infty, 0)$ because for x, $y \in X$, $$x/y > 0 \Leftrightarrow (x > 0 \wedge y > 0) \vee (x < 0 \wedge y < 0)$$
Unseen 4 marks	(iii) If $x \in \mathbb{R}$ and $x \neq 0$ then $(x, z) = x(1, z/x) \sim (1, z/x)$. Also $(1, y_1) \sim (1, y_2) \Leftrightarrow (1, y_1) = (\lambda, \lambda y_2) \Leftrightarrow \lambda = 1 \wedge y_1 = y_2$. So for each $y \in \mathbb{R}$, $(1, y)$ is in a different equivalence class, and (x, z) is in the same equivalence class as $(1, z/x)$ provided $x \neq 0$. If $x = 0$ and $z \neq 0$ then $(0, z) = z(0, 1)$. So the last equivalence class is that of $(0, 1)$.

Standard (harder) homework exercise 4 marks	10(i) . *Base case* $\frac{1}{3} < x_0 = 1$. So $\frac{1}{3} < x_n \le 1$ is true for $n = 0$. *Inductive step* Now fix $n \in \mathbb{N}$ and suppose that $\frac{1}{3} < x_n \le 1$. Then $\frac{1}{9} < x_n^2 \le 1$ and $$\frac{4}{3} + \frac{1}{9} < 1 + x_n + x_n^2 \le 3$$ So $$\frac{1}{3} < x_{n+1} = \frac{1 + x_n + x_n^2}{4} \le \frac{3}{4} < 1.$$ So $\frac{1}{3} < x_n \le 1 \Rightarrow \frac{1}{3} < x_{n+1} < 1$. So by induction $\frac{1}{3} < x_n \le 1$ holds for all $n \in \mathbb{N}$.
Calculation 2 marks	(ii) $$x_{n+2} - x_{n+1} = \frac{1 + x_{n+1} + x_{n+1}^2 - 1 - x_n - x_n^2}{4} = \frac{x_{n+1} - x_n + x_{n+1}^2 - x_n^2}{4}$$ $$= \frac{(1 + x_n + x_{n+1})(x_{n+1} - x_n)}{4}.$$
Some similarities with exercises set 4 marks	We have $x_1 = \frac{3}{4} < x_0$ and hence $x_1 - x_0 < 0$. Since $x_n \ge \frac{1}{3}$ for all n, we have $1 + x_n + x_{n+1} > 0$ for all $n \in \mathbb{N}$. So $x_{n+1} - x_n < 0 \Rightarrow x_{n+2} - x_{n+1} < 0$ and since the base case $n = 0$ holds, we have $x_{n+1} - x_n < 0$ for all $n \le \mathbb{N}$ and x_n is a decreasing sequence.
Standard homework problem on induction. 5 marks	(iii) *Base case* $$\|x_1 - x_0\| = \left\|1 - \frac{3}{4}\right\| = \frac{1}{4}$$ So the required upper bound on $\|x_{n+1} - x_n\|$ holds for $n = 0$. *Inductive step* Now suppose that the required upper bound holds on $\|x_{n+1} - x_n\|$. Then we use the formula for $\|x_{n+2} - x_{n+1}\|$ at the start of (ii). We also use the bounds $0 < x_n \le 1$ and $0 < x_{n+1} < 1$ to deduce $$\frac{1 + x_n + x_{n+1}}{2} \le 3.$$ Then from (ii) we have $$\|x_{n+2} - x_{n+1}\| = \frac{\|x_{n+1} - x_n\|(1 + x_n + x_{n+1}}{4} \le \frac{3}{4}\|x_{n+1} - x_n\|$$ $$\le \frac{3}{4} \cdot \left(\frac{3}{4}\right)^n \cdot \frac{1}{4} = \left(\frac{3}{4}\right)^{n+1} \cdot \frac{1}{4}.$$ So the upper bound for $\|x_{n+1} - x_n\|$ implies the upper bound for $\|x_{n+2} - x_{n+1}\|$, and by induction we have $$\|x_{n+1} - x_n\| \le \frac{1}{4}\left(\frac{3}{4}\right)^n$$ for all $n \in \mathbb{N}$.
15 marks in total	

Theory from lectures
5 marks

11. A set $A \subset Q$ is a *Dedekind cut* if

(i) A is nonempty, and bounded above,

(ii) $x \in A \wedge y \in \mathbb{Q} \wedge y < x \Rightarrow y \in A$

(iii) A does not have a maximal element.

1 mark

A Dedekind cut A is *rational* if it is of the form $\{x \in \mathbb{Q} : x < q\}$ for some $q \in \mathbb{Q}$. This is the Dedekind cut representing q.

Similar to homework exercises
1 mark

a) If $\{x \in \mathbb{Q} : -2 < x < \frac{1}{3}\}$ then $-2 \notin A$ and $-1 \in A$, which violates property (ii), and A is not a Dedekind cut.

1 mark

(b) If $A = \{x \in \mathbb{Q} : x > \frac{1}{3}\}$ then A is not bounded above – because $\mathbb{Z}_+ \subset A$, for example – which violates property (i), and A is not a Dedekind cut.

2 marks

c)

$$A = \left\{x \in \mathbb{Q} : x^2 + 2x + 3 < 0\right\} = \{x \in \mathbb{Q} : (x+1)^2 + 2 < 0\} = \emptyset.$$

So $A = \emptyset$, and property (i) is violated, and A is not a Dedekind cut .

Theory from lectures, but basically unseen: not expected to repeat from memory.
5 marks

Let A be a Dedekind cut, and define

$$B = \{-x : x \in \mathbb{Q} \wedge x \notin A\}$$

$B \neq \emptyset$ because $-x \in B$ for any $x \in \mathbb{Q} \setminus A$. There is at least one such x because A is bounded above.
If $y \in B$ and $z \in A$ then $z < -y$ because if $-y \leq z$ then by property (ii) for Awe have $-y \in A$ and $-(-y) \notin B$. So for any $z \in A$ we have $y < -z$ for all $y \in B$, and so B is bounded above by $-z$ for any $z \in A$.

15 marks in total

Theory from lectures
4 marks

12. A is *finite* if either A is empty, or, for some $n \in \mathbb{Z}_+$, there is a bijection $f : \{k \in \mathbb{N} : k < n\} \to A$. For a fixed set A, there is at most one $n \in \mathbb{Z}_+$ for which such a bijection exists, and if there is such an n then A is said to be *of cardinality* n. The empty set is said to be of cardinality 0.
A is *countable* if either A is finite or there is a bijection $f : \mathbb{N} \to A$.

Standard examples
4 marks

A is countable, B is uncountable, C is countable and D is uncountable.

2 marks

$f(x) = e^x$ will do, but of course there are many other examples. The inverse function is $\ln : (0, \infty) \to \mathbb{R}$.

Example from lectures but not expected to be done from memory.
5 marks

Define $g : \mathbb{N} \to \mathbb{Z}$ by

$$g(n) = \begin{cases} n/2 \text{ if n is even,} \\ -(n+1)/2 \text{ if } n \text{ is odd.} \end{cases}$$

Now

$$n/2 = p \Leftrightarrow n = 2p,$$

and

$$-(n+1)/2 = p \Leftrightarrow -n - 1 = 2p \Leftrightarrow n = -2p - 1.$$

So if we define $h : \mathbb{Z} \to \mathbb{N}$ by

$$h(p) = \begin{cases} 2p \text{ if } p \geq 0, \\ -2p - 1 \text{ if } p < 0, \end{cases}$$

then $g(n) = p \Leftrightarrow h(p) = n$ and hence $g(h(p)) = p$ for all $p \in \mathbb{Z}$ and $h(g(n)) = n$ for all $n \in \mathbb{N}$.

15 marks in total