Solutions to MATH105 exam January 2014
Section A

1 mark 2 marks 2 marks 2 marks Standard homework exercises 7 marks in total. No reasons required.	1a) $2 < 3$ or $5 < 4$. This is true. b) $2 > 3$ and $4 \leq 5$. This is false. c) If x is rational then x is an integer. This is false. For example, 0.5 is rational, but not an integer. d) There exists a real number x such that $x^2 + 2x + 1 < 0$. This is false, because $x^2 + 2x + 1 = (x + 1)^2 \geq 0$.
1 mark 1 mark 2 marks 2 marks Standard homework exercises. 6 marks in total.	2a) $2 \geq 3 \wedge 5 \geq 4$. b) $2 \leq 3 \vee 4 > 5$. c) $\exists x \in \mathbb{Q}$, $x \notin \mathbb{Z}$. d) $\forall x \in \mathbb{R}$, $x^2 + 2x + 1 \geq 0$.
1 mark 4 marks 1 mark Standard homework exercise. 6 marks in total.	3. **Base case**: When $n = 0$, $1 < \frac{3}{2} = x_0 < 2$, so $1 < x_0 < 2$ is true when $n = 0$. **Inductive step**: Suppose that $n \geq 0$ and $1 < x_n < 2$. Then $$\frac{2}{3} + \frac{1}{3} < \frac{2}{3}x_n + \frac{1}{3} = x_{n+1} < \frac{4}{3} + \frac{1}{3} = \frac{5}{3} < 2.$$ So if $n \geq 0$, $1 < x_n < 2 \Rightarrow 1 < x_{n+1} < 2$. So, by induction, $1 < x_n < 2$ for all integers $n \geq 0$.
2 marks 4 marks Bookwork 6 marks in total	4. If m and n are integers, then m *divides* n if $n = mk$ for some integer k. If m, n and p are all integers, then $n = mk_1$ and $p = nk_2$ for integers k_1 and k_2. So $p = nk_2 = mk_1k_2 = m(k_1k_2)$. Then since k_1k_2 is an integer, m divides p.

1 mark — 5a) $((0,2] \cup [1,3]) \cup [2,4] = (0,3] \cup [2,4] = (0,4]$

1 mark — b) $(0,2] \cap [1,3]) \cup [2,4] = [1,2] \cup [2,4] = [1,4]$.

1 mark — c) $((0,2] \cap [1,3]) \setminus [2,4] = [1,2] \setminus [2,4] = [1,2)$.

Standard homework exercises.
3 marks in total.

6.

$$\begin{array}{cc|c} 1 & 0 & 1014 \\ 0 & 1 & 455 \end{array} \quad \overset{R_1 - 2R_2}{\rightarrow} \quad \begin{array}{cc|c} 1 & -2 & 104 \\ 0 & 1 & 455 \end{array} \quad \underset{R_2 - 4R_1}{\rightarrow} \quad \begin{array}{cc|c} 1 & -2 & 104 \\ -4 & 9 & 39 \end{array}$$

$$\overset{R_1 - 2R_2}{\rightarrow} \quad \begin{array}{cc|c} 9 & -20 & 26 \\ -4 & 9 & 39 \end{array} \quad \underset{R_2 - R_1}{\rightarrow} \quad \begin{array}{cc|c} 9 & -20 & 26 \\ -13 & 29 & 13 \end{array}$$

$$\overset{R_1 - 2R_2}{\rightarrow} \quad \begin{array}{cc|c} 35 & -78 & 0 \\ -13 & 29 & 13 \end{array}$$

4 marks

As a result of this:

1 mark — (i) the g.c.d. d is 13;

1 mark — (ii) from the first row of the last matrix, $m_1 = 78$ and $n_1 = 35$;

1 mark — (iii) from the second row of either of the last two matrices $a = -13$ and $b = 29$;

2 marks — (iv) The l.c.m. is $1014 \times 35 = 35490$.

Standard homework exercise.
9 marks in total

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1 mark — 7. The image of f, $\mathrm{Im}(f)$, is $\{f(x) : x \in X\}$.

2 marks — a) $y = 2x + 1 \Leftrightarrow x = (y-1)/2$. So in this case every real number y is in the image, and $\mathrm{Im}(f) = \mathbb{R}$.

2 marks — b) We have $0 \leq (y-1)/2 \leq 1 \Leftrightarrow 0 \leq y - 1 \leq 2 \Leftrightarrow 1 \leq y \leq 3$. So in this case, with $X = [0,1]$, we have $\mathrm{Im}(f) = [1,3]$.

3 marks — By definition, $\mathrm{Im}(g \circ f) = \{g(f(x)) : x \in X\}$. Since $f(x) \in Y$ for all $x \in X$, we see that $\mathrm{Im}(g \circ f) \subset \{g(y) : y \in Y\} = \mathrm{Im}(g \circ f)$.

Standard theory followed by two standard homework exercise and bookwork which was set in homework.
8 marks in total.

Standard theory. 2 marks 2 marks Standard theory followed by standard homework exercise 4 marks	8. A real number x is *algebraic* if there are $n \in N$ and integers a_i, for $0 \le i \le n$, such that $\sum_{i=0}^{n} a_i x^i = 0$. If $x = 1 + \sqrt{2}$ then $(x-1)^2 = 2$, that is, $x^2 - 2x - 1 = 0$.
1 mark 1 mark 1 mark 1 mark 2 marks Bookwork followed by standard homework exercises. 6 marks	9. $f : X \to Y$ is *injective* if, for any x_1 and $x_2 \in X$, $f(x_1) = f(x_2) \;\Rightarrow\; x_1 = x_2$ A is countable if A is empty or there is an injective map $f : A \to \mathbb{N}$. We can also take the codomain to be $\mathbb{Z}$ or $\mathbb{Z}_+$. a) Uncountable. b) Countable. c) Countable.

Section B

Theory from lectures
3 marks

10. $\sim$ is *reflexive* if

$$x \sim x \forall x \in X$$

$\sim$ is *symmetric* if

$$x \sim y \;\Rightarrow\; y \sim x \;\forall\; x,\; y \in X.$$

$\sim$ is *transitive* if

$$(x \sim y \wedge y \sim z) \;\Rightarrow\; x \sim z \;\forall\; x,\; y,\; \in X.$$

Theory from lectures.
2 marks

The equivalence class $[x]$ of x is the set $\{y \in X : y \sim x\}$.

Standard homework exercise
2 marks

a) $n \mid n$ for all integers n. So $\sim$ is *reflexive.* However $1 \mid 2$ and $2 \not| 1$ so $\sim$ is not symmetric and not an equivalence relation.

Standard homework exercise.
3 marks

b) 5 divides $0 = n - n$ for any $n \in \mathbb{Z}$. So $\sim$ is *reflexive* If $5 \mid m - n$ then $m - n = 5k$ for some $k \in \mathbb{Z}$ and $n - m = 5(-k)$ and sinc e$-k \in \mathbb{Z}$ we have $5 \mid (n - m)$. So $m \sim n \Rightarrow n \sim m$ and $\sim$ is *symmetric.* If $5 \mid (m - n)$ and $5 \mid (n - p)$ when $m - n = 5k_1$ amd $n - p = 5k_2$ for some $k_1,\ k_2 \in \mathbb{Z}$, and $m - p = 5(k_1 + k_2$, and $k_1 + k_2 \in \mathbb{Z}$, and $5 \mid (m - p)$. So $(m \sim n \wedge n \sim p) \Rightarrow m \sim p$ and $\sim$ is *transitive.* So $\sim$ is an equivalence relation.

Standard exercise, with notation likely to prove more challenging
3 marks

c) If $f(x)$ is any polynomial, $f(0) = f(0)$. So $\sim$ is *reflexive.* If $f(x)$ and $g(x)$ are any polynomials and $f(0) = g(0)$, then $g(0) = f(0)$. So $f(x) \sim g(x) \Rightarrow g(x) \sim f(x)$ and $\sim$ is *symmetric.* If $f(x)$, $g(x)$ and $h(x)$ are any three polynomials, and $f(0) = g(0)$ and $g(0) = h(0)$, then $f(0) = h(0)$. So $(f(x) \sim g(x) \wedge g(x) \sim h(x)) \Rightarrow f(x) \sim h(x)$ and $\sim$ is *transitive.* So $\sim$ is an equivalence relation.

Harder exercise, not previously set.
2 marks

Write $f(x) = x^2 - x$. Then $f(0) = 0$. So $[f(x)] = \{h(x) : h(0) = 0.$ Now if $h(x) = \sum_{i=0}^{n} a_i x^n$ then

$$h(0) = 0 \;\Leftrightarrow\; a_0 = 0 \;\Leftrightarrow\; (h(x) = x \sum_{i=1}^{n} a_i x^{i-1} \vee (n = 0 \wedge h(x) = 0)$$

$$\Leftrightarrow h(x) = xg(x) \text{ for some polynomial } g(x).$$

15 marks in total.

Standard homework exercise 1 mark	11a) *Base case*: $2^2 - 1 = 3$. So $n < 2^n - 1$ holds for $n = 2$.
4 marks	*Inductive step*: Suppose that for some $n \in \mathbb{Z}_+$ we have $n < 2^n - 1$. Then, for all $n \geq 2$, $$2^{n+1} - 1 = 2 \times 2^n - 1 > 2n - 1 = n + n - 1 \geq n + 1$$ So $n < 2^n - 1 \Rightarrow n + 1 < 2^{n+1} - 1$
1 mark Theory from lectures	So by induction $n < 2^n - 1$ is true for all integers $n \geq 2$.
Unseen, but similar to a step in 2 proofs from lectures. 3 marks	b) $\prod_{i=1}^n p_i$ is divisible by p_i for all $1 \leq i \leq n$, and hence $1 + \prod_{i=1}^n p_i$ is not divisible by p_i for any $1 \leq i \leq n$. But by the Fundamental Theorem of Arithmetic, it must be divisible by some prime p_i with $i \geq n + 1$. So $p_{n+1} \leq 1 + \prod_{i=1}^n p_i$.
Unseen but with some similarity to some homework exercises. 1 mark	*Base case*: $1 + \prod_{i=1}^1 p_i = 2 < 3 = 4 - 1 = 2^{2^1} - 1$, so the base case $n = 1$ is true.
4 marks	*Inductive step* Fix $n \in \mathbb{Z}_+$ and suppose that $\prod_{i=1}^n p_i < 2^{2^n} - 1$. Then $p_{n+1} \leq 1 + \prod_{i=1}^n p_i < 2^{2^n}$. So $$\prod_{i=1}^{n+1} p_i = \prod_{i=1}^{n} p_i \times p_{n+1} < (2^{2^n} - 1) \times 2^{2^n} = 2^{2^n} \times 2^{2^n} - 2^{2^n} \leq 2^{2^{n+1}} - 4 < 2^{2^{n+1}} - 1.$$
1 mark	So true for n implies true for $n+1$, and, by induction, $\prod_{i=1}^n p_i < 2^{2^n} - 1$ for all $n \in \mathbb{Z}_+$
15 marks in total	

Bookwork 4 marks	12 $f : X \to Y$ is *injective* if, for all x_1 and $x_2 \in X$, $$f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$ $f : X \to Y$ is *surjective* if $\text{Im}(f) = Y$ where $\text{Im}(f)$ is defined to be the set $\{f(x) : x \in X$. $f : X \to Y$ is *a bijection* if it is both injective and surjective.
Bookwork, some similar exercises 3 marks	Suppose that $f : X \to Y$ and $g : Y \to Z$ are both injective and suppose that x_1 and $x_2 \in X$ and $g \circ f(x_1) = g \circ f(x_2)$, that is, $g(f(x_1)) = g(f(x_2))$. Then since g is injective we have $f(x_1) = f(x_2)$, and since f is injective, we have $x_1 = x_2$. So $g \circ f$ is injective.
Bookwork 3 marks	*Schröder Bernstein Theorem* Suppose that X and Y are sets and there are injective maps $f : X \to Y$ and $g : y \to X$. Then there is a bijection $h : X \to Y$.
Standard exercise 3 marks	$f : (0, 1] \to [0, 1]$ is injective where $f(x) = x$, for all $x \in (0, 1]$. Also, $g : [0, 1] \to (0, 1]$ is injective, where $g(x) = (x + 1)/2$ for all $x \in [0, 1]$. Note that $\frac{1}{2} \leq g(x) \leq 1$ for all $x \in [0, 1]$. So by the Schröder Bernstein theorem, there is a bijection between $(0, 1]$ and $[0, 1]$.
2 marks	Suppose that A_n is countable for all n. Then there is an injective map $f_n : A_n \to \mathbb{Z}_+$. Let $g : \mathbb{Z}_+^2 \to \mathbb{Z}_+$ be a bijection. Then define $h(x) = g(f_n(x), n)$ for $x \in A_n$, for each $n \in \mathbb{Z}_+$. Then $h : \cup_{n=1}^{\infty} A_n \to \mathbb{Z}_+$ is injective
15 marks in total.	

Theory from lectures
4 marks

13. A set $A \subset \mathbb{Q}$ is a *Dedekind cut* if

(i) $A \neq \emptyset$

(ii) $\mathbb{Q} \setminus A \neq \emptyset$

(iii) $x \in A \wedge y \in \mathbb{Q} \wedge y < x \Rightarrow y \in A$;

(iv) A does not have a maximal element.

Similar to homework exercises.

1 mark

a) $6 \in A$ but $4 \notin A$. So property (iii) does not hold and A is not a Dedekind cut

3 marks

b) $0 \in A$ and $6 \notin A$, so properties (i) and (ii) hold. If $x < 5$ and $y \in \mathbb{Q}$ and $y < x$ then $y < 5$, so property (iii) holds. Finally A does not have a maximal element. For suppose $a \in A$. Then $a < 5$ and hence $a < (a+5)/2 < 5$. So if $a \in A$ we also have $(a+5)/2 \in A$, and a cannot be maximal in A. So A is a Dedekind cut

5 marks

c) $0 \in A$ and $3 \notin A$. So properties (i) and (ii) hold for A. Now let $x \in A$ and let $y \in \mathbb{Q}$ and $y < x$. If $y < 0$ the $y \in A$. If $0 \leq y < x$ then $0 \leq y^2 < x^2 < 5$, and, once again, $y \in A$. So property (iii) holds for A. Now suppose $a \in A$. If $a < 2$ the x is not maximal because $2 \in A$. So suppose $a \geq 2$. We also have $a < 3$ because if $a \geq 3$ then $a^2 \geq 9 > 5$. Now if $\varepsilon \in \mathbb{Q}$ and $1 > \varepsilon > 0$ then $(a+\varepsilon)^2 = a^2 + 2a\varepsilon + \varepsilon^2 < a^2 + 7\varepsilon$. So if $0 < \varepsilon \leq (5 - a^2)/8$ we see that $(a+\varepsilon)^2 < 5$ and a is not maximal in A. So property (iv) holds for A and A is a Dedekind cut.

2 marks

(iv) $0 \in A$ but $-3 \notin A$ so property (iii) is violated and A is not a Dedekind cut.

15 marks in total