Solutions to MATH105 exam January 2013
Section A

1 mark	1.a) 1.5 is an integer This is false.
2 marks	b) If x is a real number and $x^2 \leq 0$, then $x = 0$ This is true, because if x is a non-zero real number, $x^2 > 0$.
2 marks	c) For any integer n, 2 divides n or $n - 1$. This is true.
2 marks	d) $0 \leq x \leq 1$ if and only if $0 \leq x^2 \leq 1$: false because $0 \leq (-1)^2 \leq 1$ (for example).
Standard homework exercises 7 marks in total. No reasons required.	
2 marks	2a)$x^2 < 9 \Leftrightarrow -3 < x < 3$. So $\{x : x^2 < 9\} = (-3, 3)$.
4 marks	b) For $\dfrac{x}{x+2} > 3$ either both x and $x+2$ have to be > 0 or both < 0. If they are both positive we must have, in addition, $x > 3x+6$, that is, $x < -3$, which is inconsistent with $x > 0$. If both are negative, we must have, in addition, $x < 3x+6$, that is, $x > -3$. So we must have $-3 < x < -2$, that is, $\left\{x \in \mathbb{R} : \dfrac{x}{x+2} > 3\right\} = (-3, -2).$
Standard homework exercises. 6 marks in total.	
1 mark	3. **Base case**: When $n = 3$ both n^3 and 3^n are 3^3, so $n^3 \leq 3^n$ is true when $n = 3$
4 marks	**Inductive step**: Suppose that $n \geq 3$ and $n^3 \leq 3^n$. Then $(n+1)^3 = n^3 \times (1+\frac{1}{n})^3 \leq n^3 \times (\frac{4}{3})^3 = \frac{64}{27}n^3 \leq 2n^3 \leq 2 \times 3^n < 3^{n+1}.$ So if $n \geq 3$, $n^3 \leq 3^n \Rightarrow (n+1)^3 \leq 3^{n+1}$.
1 mark	So, by induction, $n^3 \leq 3^n$ for all integers $n \geq 3$
Standard homework exercise. 6 marks in total.	

6 marks	4. $1989 = 9 \times 221 = 3^2 \times 13 \times 17$. So the divisors of 1989 are $3^{k_1} \times 13^{k_2} \times 17^{k_3}$ for integers $0 \le k_1 \le 2$, $0 \le k_2 \le 1$, $0 \le k_3 \le 2$, that is, 1, 3, 9, 13, 39, 117, 17, 51 153, 221, 663, 1989.
Standard homework exercises. 6 marks in total.	
2 marks	5. $n \in \mathbb{Z}$ is even $\Leftrightarrow n = 2k$ for some $k \in \mathbb{Z} \Rightarrow n^2 = 4k^2 = 2(2k)^2$ $\Rightarrow n^2$ is even.
3 marks	n not even $\Rightarrow n - 1$ is even $\Rightarrow n - 1 = 2k$ for some $k \in \mathbb{Z} \Rightarrow$ $n = 2k + 1 \Rightarrow n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 \Rightarrow$ n^2 is not even.
Bookwork. 5 marks in total.	
4 marks	6. $\begin{array}{rr\|r} 1 & 0 & 623 \\ 0 & 1 & 231 \end{array} \xrightarrow{R_1 - 2R_2} \begin{array}{rr\|r} 1 & -2 & 161 \\ 0 & 1 & 231 \end{array} \xrightarrow[R_2 - R_1]{} \begin{array}{rr\|r} 1 & -2 & 161 \\ -1 & 3 & 70 \end{array}$ $\xrightarrow{R_1 - 2R_2} \begin{array}{rr\|r} 3 & -8 & 21 \\ -1 & 3 & 70 \end{array} \xrightarrow[R_2 - 3R_1]{} \begin{array}{rr\|r} 3 & -8 & 21 \\ -10 & 27 & 7 \end{array}$ $\xrightarrow{R_1 - 3R_2} \begin{array}{rr\|r} 33 & -89 & 0 \\ -10 & 27 & 7 \end{array}$
	As a result of this:
1 mark	(i) the g.c.d. d is 7;
1 mark	(ii) from the first row of the last matrix, $m_1 = 89$ and $n_1 = 33$;
1 mark	(iii) from the second row of either of the last two matrices $a = -10$ and $b = 27$;
2 marks	(iv) The l.c.m. is $623 \times 33 = 20559$.
Standard homework exercise. 9 marks in total	

2 marks 2 marks 4 marks Standard theory followed by standard homework exercise. 8 marks in total.	7 $f : X \to Y$ is injective if $\forall x_1,\ x_2 \in X,\ f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$. The image of f, $\mathrm{Im}(f)$ is $\{f(x) : x \in X\}$. f is a *bijection* if f is injective and $\mathrm{Im}(f) = Y$, that is, f is also surjective $y = \dfrac{x+1}{x+2} \Leftrightarrow x+1 = y(x+2) \Leftrightarrow x(1-y) = 2y-1 \Leftrightarrow x = \dfrac{2y-1}{1-y}$. It follows that f is injective. Also, we see that $x > 0 \Leftrightarrow \frac{1}{2} < y < 1$. So $\mathrm{Im}(f) = (\frac{1}{2}, 1)$.
Standard theory. 2 marks 3 marks Standard homework exercises. 5 marks	8. (i) $\dfrac{x+2}{x+3} = 10 \Leftrightarrow x+2 = 10x+30 \Leftrightarrow x = -\dfrac{28}{9}$. So x is rational. (ii) $\dfrac{2}{y+2} = 1-\dfrac{1}{y+1} \Leftrightarrow 2(y+1) = (y+1)(y+2)-(y+2) \Leftrightarrow 2y+2 = y^2+3y+2-y-$ So y is the positive square root of 2, which is not rational.
1 mark 1 mark 1 mark Standard homework exercises. 3 marks	9(i) countable; (ii) uncountable; (iii) countable.

Section B

Theory from lectures 3 marks	10. $\sim$ is *reflexive* if $$x \sim x \forall x \in X$$ $\sim$ is *symmetric* if $$x \sim y \;\Rightarrow\; y \sim x \;\forall\; x,\; y \in X.$$ $\sim$ is *transitive* if $$(x \sim y \wedge y \sim z) \;\Rightarrow\; x \sim z \;\forall\; x,\; y,\; \in X.$$
Theory from lectures. 2 marks	The equivalence class $[x]$ of x is the set $\{y \in X : y \sim x\}$.
Standard homework exercise 3 marks	(i) $n - n = 0 = 3 \times 0$. So $n \sim n \; \forall n \in \mathbb{Z}$ and $\sim$ is *reflexive*. If $m \sim n$ then $m - n = 3r$ for $r \in \mathbb{Z}$, and hence $n - m = 3(-r)$ for $-r \in \mathbb{Z}$ and $n \sim m$. So $\sim$ is *symmetric* If $m \sim n$ and $n \sim p$, then $m - n = 3r$ and $n - p = 3s$ for some $r,\ s \in \mathbb{Z}$, and hence $m - p = m - n + (n - p) = 3(r + s)$. So $\sim$ is *transitive* and $\sim$ is an equivalence relation.
Standard homework exercise. 1 mark	(ii) For any $n \in \mathbb{Z}$, $n - n = 0 \neq 3k + 1$ for any $k \in \mathbb{Z}$. So $\sim$ is not reflexive and is not an equivalence relation.
Harder exercise, not previously set. 4 marks	(iii) $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \sim \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ because $c_1 = c_1$. So $\sim$ is reflexive. Since $c_1 = c_2 \Leftrightarrow c_2 = c_1$, we have $$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \sim \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \sim \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$ and $\sim$ is symmetric. Since $c_1 = c_2 \wedge c_2 = c_3 \Rightarrow c_1 = c_3$, we have: $$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \sim \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \wedge \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \sim \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \sim \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix},$$ and $\sim$ is transitive.
Harder exercise, not previously set. 2 marks	The equivalence class of $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is $$\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{R} \right\}$$
15 marks in total.	

Standard homework exercise 1 mark	11(i) *Base case*: $1 = 1 \times (1+1)/2$. So the formula holds for $n = 1$
4 marks	*Inductive step*: Suppose that for some $n \in \mathbb{Z}_+$ we have $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. Then $$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + n + 1 = \frac{n(n+1)}{2} + n + 1 = (n+1)\left(\frac{n}{2} + 1\right) = \frac{(n+1)(n+2)}{2} = \frac{(n+1)(n+1+1)}{2}$$ So $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \Rightarrow \quad \sum_{k=1}^{n+1} k = \frac{(n+1)(n+1+1)}{2}$$
1 mark	So by induction $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ holds for all $n \in \mathbb{Z}_+$.
Unseen: corresponding case of even and odd integers proved in lectures as an example of induction. 1 mark	(ii) *Base case*: 0 is divisible by 3. So the statement is true for $n = 0$.
4 marks	*Inductive step*: Suppose that exactly one of n, $n-1$ and $n+1$ is divisible by 3. Since $n+2 = (n-1)+3$ is divisible by 3 if and only if $n-1$ is, exactly one of n, $n-1$ and $n+1$ is divisible by 3 if and only if exactly one of $n = (n+1)-1$ and $n+2 = (n+1)+1$ and $n+1$ is divisible by 3. So if the statement is true for n, it is true for $n+1$.
1 mark	So by induction, for all $n \in \mathbb{N}$, exactly one of n, $n-1$ and $n+1$ is divisible by 3.
3 marks	If $n \in \mathbb{N}$ is divisible by 3 then $n = 3k$ for some $k \in \mathbb{N}$. If $n \in \mathbb{N}$ and $n+1$ is divisible by 3 then $n+1 = 3k$ for some $k \in \mathbb{Z}_+ \subset N$ and $n = 3k-1$ for some $k \in \mathbb{Z}_+$. If $n-1$ is divisible by 3 and $n \in \mathbb{N}$ then $n > 0$ and $n-1 = 3k$ for some $k \in N$, and $n = 3k+1$ for some $k \in \mathbb{N}$.
15 marks in total	

Theory from lectures 2 marks	12(i) The inclusion/exclusion principle to two finite sets A and B is that $$\|A \cup B\| = \|A\| + \|B\| - \|A \cap B\|.$$
Standard homework exercises. 2 marks	(ii) If B is the set of people who bought bread and A is the set of people who bought another item, then $\|B\| = 66$, $\|A\| = 128\|$ and $\|A \cup B\| = 150$. The set of people who bought both bread and another item is $A \cap B$, and from (i) we have $$\|A \cap B\| = \|A\| + \|B\| - \|A \cup B\| = 128 + 66 - 150 = 44.$$
2 marks	(iii) The set of people who bought only bread is $B \setminus A$. we have $$\|B \setminus A\| = \|B\| - \|B \cap A\| = 66 - 44 = 22.$$
1 mark	(iv) The set of people not buying bread is $A \setminus B = (A \cup B) \setminus B$, so we have $$\|A \setminus B\| = \|A \cup B\| - \|B\| = 150 - 66 = 84.$$
2 marks	(v) Let M be the set of people buying milk. Since 56 of the 150 people bought neither bread nor milk, 94 bought either bread or milk, that is $\|M \cup B\| = 94$. We are also given $\|M \cap B\| = 31$. So from (i) with $A = M$, we have $$94 = \|M \cup B\| = \|M\| + \|B\| - \|M \cap B\| = \|M\| + 66 - 31 = \|M\| + 35,$$ that is, $\|M\| = 94 - 35 = 59$, so 59 people bought milk.
2 marks	(vi) The set of people who bought milk and not bread is $M \setminus B$, and $$\|M \setminus B\| = \|M\| - \|M \cap B\| = 59 - 31 = 28.$$ So 28 people bought milk and not bread.
2 marks	(vii) In (i) we take $B \cap M$ and $B \cap O$ to replace A and B, because $B \cap (M \cup O) = (B \cap M) \cup (B \cap O)$. Also $(B \cap M) \cap (B \cap O) = B \cap M \cap O$. So this gives $$\|B \cap (M \cup O)\| = \|B \cap M\| + \|B \cap O\| - \|B \cap M \cap O\|.$$
2 marks	(viii) Given that $\|B \cap O\| = 42$, and $\|B \cap M\| = 31$, and $\|B \cap (M \cup O)\| = 44$ from (ii), we have, from (vii) $$44 = 42 + 31 - \|B \cap M \cap O\|$$ and hence $\|B \cap M \cap O\| = 73 - 44 = 29$, that is, 29 people bought bread and milk and something else.
15 marks in total.	

Theory from lectures 4 marks	13. A set $A \subset \mathbb{Q}$ is a *Dedekind cut* if a) A is nonempty, and bounded above; b) $x \in A \wedge y \in \mathbb{Q} \wedge y < x \Rightarrow y \in A$; c) A does not have a maximal element.
Similar to homework exercises.	
1 mark	(i) $A = \{x \in \mathbb{Q} : x \leq 2/3\}$ has a maximal element $2/3$. So property c) is violated and A is not a Dedekind cut.
2 marks	(ii) $1 \in A$ and $0 \notin A$ (for example). So property b) is violated and A is not a Dedekind cut.
4 marks	(iii) If $f(x) = x^3 - 3x + 3$, then $f'(x) = 3x^2 - 3$ has zeros at ± 1, and as $f''(x) = 6x$, we see that -1 is a local maximum and 1 is a local minimum, and f is strictly increasing on $(-\infty, -1]$ and on $[1, \infty)$, and strictly decreasing on $[-1, 1]$. Since $f(1) = 1 > 0$, we see that $f(x) \geq 1$ for all $x \geq -1$. So A is bounded above by -1. But $f(-3) < 0$, so $-3 \in A$ and A is nonempty. Since f is strictly increasing on $(-\infty, -1]$, if $f(x) < 0$ and $y < x$ then $f(y) < 0$, that is, property b) holds. If $x \in A$ is a maximal element in A then $f(x) < 0$. But then by continuity of f at x, there is a rational $\delta > 0$ such that for any $y \in \mathbb{Q}$ with $x \leq y \leq x + \delta$, we have $f(y) < 0$, that is $y \in A$, contradicting x being a maximal element, that is property c) holds for A. So A is a Dedekind cut.
4 marks	(iv) Once again, if $f(x) = x^3 - 3x + 1$, then $f'(x) = 3x^2 - 3$ has zeros at ± 1, -1 is a local maximum, and f is strictly increasing on $(-\infty, -1]$, with $f(-1) = 3 > 0$. By the definition of A, the set is bounded above by -1, and since $-2 \in A$, it is non-empty. As in (iii), property b) holds because f is strictly increasing on $(-\infty, -1]$ and, again as in (iii), property c) holds because of continuity of f. So A is a Dedekind cut.
15 marks in total	