Solutions to MATH105 exam January 2012
Section A

3 marks	1.a) For any real number x, $x^2 + 2x + 1 \geq 0$. This is true because $x^2+2x+1 = (x+1)^2$, and the square of a real number is always greater than or equal to 0.
3 marks	b) There exists an integer n such that $n \leq p$ for all integers p. This is clearly false, because whatever n is, if $p = n-1$ then $n > p$.
Standard homework exercises 6 marks in total	
2 marks	2a)$x \geq 0 \wedge x < 2$.
2 marks	b) $\exists x \in (0,1)$ such that $x \leq \sin x$.
Standard homework exercises 4 marks in total	
2 marks	3a) $-4 < -3 < -1$ and $6 < 7 < 8$. So $[-3,7] \cap (-1,8) \cap [-4,6] = (-1,6]$.
2 marks	b) $([-4,1] \cup (3,8]) \cap [2,3] = ([-4,1] \cap [2,3]) \cup ((3,8] \cap [2,3]) = \emptyset \cup \emptyset = \emptyset$.
2 marks	c) $[-5,1] \cap (2,6) = \emptyset = (3,7) \cap [-3,0]$. So $$([-5,1] \cup (3,7)) \cap ([-3,0] \cup (2,6)) = [-5,1] \cap [-3,0]) \cup ((3,7) \cap (2,6))$$ $$= [-3,0] \cup (3,6).$$
Standard homework exercises 6 marks in total	
4 marks	4a) $x^2 + x > 2 \ \Leftrightarrow x^2 + x - 2 > 0 \Leftrightarrow (x+2)(x-1) > 0$ $\Leftrightarrow (x+2 > 0 \wedge x-1 > 0) \vee (x+2 < 0 \wedge x-1 < 0) \Leftrightarrow x > 1 \vee x < -2$.
2 marks	b) $x^2 + x + 2 = (x + \frac{1}{2})^2 + \frac{7}{4} > 0$ for all $x \in \mathbb{R}$. So there are no solutions, that is, the set of solutions is empty.
Standard homework exercises. 6 marks in total	

Marks	Solution
1 marks	5. To start the induction, $2-3^0=2-1=1$ So $x_n=2-3^n$ holds for $n=0$. Now suppose inductively that $x_n=2-3^n$. Then
4 marks	$$x_{n+1}=3x_n-4=3(2-3^n)-4=6-3^{n+1}-4=2-3^{n+1}.$$ So true for n implies true for $n+1$ and $x_n=2-3^n$ is true for all $n\in\mathbb{N}$.
Standard homework exercise 5 marks in total	
	6.
4 marks	$$\left.\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right\rvert \begin{array}{c} 572 \\ 385 \end{array} \xrightarrow{R_1-R_2} \left.\begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array}\right\rvert \begin{array}{c} 187 \\ 385 \end{array} \xrightarrow[R_2-2R_1]{} \left.\begin{array}{cc} 1 & -1 \\ -2 & 3 \end{array}\right\rvert \begin{array}{c} 187 \\ 11 \end{array}$$ $$\xrightarrow{R_1-17R_2} \left.\begin{array}{cc} 35 & -52 \\ -2 & 3 \end{array}\right\rvert \begin{array}{c} 0 \\ 11 \end{array}$$
	As a result of this:
1 mark	(i) the g.c.d. d is 11;
1 mark	(ii) from the first row of the last matrix, $r=52$ and $s=35$;
1 mark	(iii) from the second row of either of the last two matrices $m=-2$ and $n=3$;
2 marks	(iv) The l.c.m. is $572\times 35=20020$.
Standard homework exercise 9 marks in total	
2 marks	7 $f:X\to Y$ is injective if $\forall x_1,\ x_2\in X,\ f(x_1)=f(x_2\Leftrightarrow x_1=x_2$.
3 marks	The image of f, $\mathrm{Im}(f)$ is $\{f(x):x\in X\}$. f is a *bijection* if f is injective and $\mathrm{Im}(f)=Y$, that is, f is also surjective
3 marks	a) Since f is strictly decreasing on $(0,\infty)$, it is injective. For all $x\in(0,\infty)$,we have $x^{-2}>0$, and $x^{-2}=y\Leftrightarrow x=1/\sqrt{y}$. So $\mathrm{Im}(f)=(0,\infty)$.
2 marks	b) f is not injective, since $\sin^2(-x)=(-\sin x)^2=\sin^2 x$. For all x we have $-1\le\sin x\le 1$, and $\sin([0,\pi/2])=[0,1]$. So $\mathrm{Im}(f)=[0,1]$.
Standard theory followed by standard homework exercises 10 marks in total	

standard theory 2 marks	8. $\lvert A_1 \cup A_2\rvert = \lvert A_1\rvert + \lvert A_2\rvert - \lvert A_1 \cap A_2\rvert$
standard homework exercise 3 marks	(i) If A_1 and A_2 are the sets of retailers selling Series 1 and 2 respectively, then $\lvert A_1 \cup A_2\rvert = 10$ and $\lvert A_1\rvert = 9$ and $\lvert A_2\rvert = 8$, then the inclusion/exclusion principle gives $\lvert A_1 \cap A_2\rvert = 9 + 8 - 10 = 7$. Then the number of retailers selling only Series 1 is $\lvert A_1\rvert - \lvert A_1 \cap A_2\rvert = 9 - 7 = 2$ and the number selling only Series 2 is $\lvert A_2\rvert - \lvert A_1 \cap A_2\rvert = 8 - 7 = 1$
unseen 4 marks Part marks will be given for an answer which recognises some possibilities without giving the general solution. Standard homework exercise 9 marks in total	(ii) Let A_3 denote the set of retailers selling Series 3. Since this is included in the original set of 10 retailers, we have $A_3 \subset A_1 \cup A_2$, and every retailer which sells Series 3 also sells at least one of Series 1 and 2. So if 6 of the retailers sell all three, there is one retailer who sells Series 3 and exactly one other of Series 1 and 2. There are 7 retailers who sell both Series 1 and Series 2, but only 6 of these sell Series 3. So there is one retailer who sells just Series 1 and Series 2, and one other who sells Series 3 and just one other. So altogether 2 retailers sell exactly 2 of the 3 series.

Section B

Theory from lectures
3 marks

9. $\sim$ is *reflexive* if

$$x \sim x \forall x \in X$$

$\sim$ is *symmetric* if

$$x \sim y \;\Rightarrow\; y \sim x \;\forall\; x,\; y \in X.$$

$\sim$ is *transitive* if

$$(x \sim y \wedge y \sim z) \;\Rightarrow\; x \sim z \;\forall\; x,\; y,\; \in X.$$

Standard homework exercise
3 marks

(i) $n - n = 0 \in \mathbb{Z}$ is even. So $n \sim n \;\forall n \in \mathbb{Z}$ and $\sim$ is *reflexive* If $m \sim n$ then $m - n = 2r$ for $r \in \mathbb{Z}$, and hence $n - m = 2(-r)$ is even and $n \sim m$. So $\sim$ is *symmetric* If $m \sim n$ and $n \sim p$, then $m - n = 2r$ and $n - p = 2s$ for some $r,\; s \in \mathbb{Z}$, and hence $m - p = m - n + (n - p) = 2(r + s)$ is even. So $\sim$ is *transitive* and $\sim$ is an equivalence relation

Standard homework exercise
2 marks

For any $m \in \mathbb{Z}$, either $m = 2p$ for some $p \in \mathbb{Z}$ or $m = 2q - 1$ for some $q \in \mathbb{Z}$ —but not both. So either $m \sim 0$ or $m \sim 1$ – but not both. So there are two equivalence classes, and 0 and 1 are representatives.

Standard homework exercise
4 marks

(ii) $f(x) - f(x) = 0 = 0 + 0x$. So $f \sim f \forall f \in X$, and $\sim$ is reflexive. Now suppose that $f(x) - g(x) = \alpha_0 + \alpha_1 x$ where α_0 and α_1 are even. Then $g(x) - f(x) = -\alpha_0 - \alpha_1 x - x^2 F(x)$ and $-\alpha_0$ and $-\alpha_1$ are even. So $f \sim g \Rightarrow g \sim f$ and $\sim$ is symmetric.
Now suppose also that $g(x) - h(x) = \beta_0 + \beta_1 x$ where β_0 and β_1 are even. Then

$$f(x) - h(x) = \alpha_0 + \beta_0 + (\alpha_1 + \beta_1)x$$

where $\alpha_0 + \beta_0$ and $\alpha_1 + \beta_1$ are even. So

$$f \sim g \wedge g \sim h \Rightarrow f \sim h$$

and $\sim$ is transitive.
So $\sim$ is an equivalence relation.

Standard exercise not previously set
3 marks

Representatives of the four equivalence classes are

$$0,\; 1,\; x,\; x + 1.$$

because if $f(x) = c_0 + c_1 x$ for $c_0,\, c_1 \in \mathbb{Z}$ then $c_0 = 2d_0$ or $1 + 2d_0$ for $d_0 \in \mathbb{Z}$ – but not both – and $c_1 = 2d_1$ or $1 + 2d_1$ for $d_1 \in \mathbb{Z}$ – but not both – and hence exactly one of the following holds

$$f(x) \sim 0, \quad f(x) \sim 1, \quad f(x) \sim x, \quad f(x) \sim 1 + x.$$

15 marks in total

Standard (harder) homework exercise
4 marks

10(i) . *Base case* $1 = x_0 < 2$. So $1 \leq x_n < 2$ is true for $n = 0$.
Inductive step Now fix $n \in \mathbb{N}$ and suppose that $1 \leq x_n < 2$. Then $4 \leq 3 + x_n < 5$ and

$$1 < \frac{7}{5} < \frac{7}{3+x_n} \leq \frac{7}{4} < 2.$$

So

$$1 = 3 - 2 \leq x_{n+1} = 3 - \frac{7}{3+x_n} < 3 - 1 = 2.$$

So $1 \leq x_n < 2 \;\Rightarrow\; 1 < x_{n+1} < 2$.
So by induction $1 \leq x_n < 2$ holds for all $n \in \mathbb{N}$.

Calculation
2 marks

(ii)

$$x_{n+2} - x_{n+1} = 3 - \frac{7}{3+x_{n+1}} - 3 + \frac{7}{3+x_n} = \frac{7(3 + x_{n+1} - 3 - x_n)}{(3+x_{n+1})(3+x_n)}$$

$$= \frac{7(x_{n+1} - x_n)}{(3+x_n)(3+x_{n+1})}.$$

Some similarities with exercises set
4 marks

The denominator of the expression on the right-hand side is > 0 by (i), because $x_n > 0$ and $x_{n+1} > 0$. So $x_n < x_{n+1} \Rightarrow x_{n+1} < x_{n+2}$. We have $x_0 < x_1 = \frac{5}{4}$. So the base case of $x_n < x_{n+1}$ for $n = 0$ holds and the inductive step has just been proved. So by induction $x_n < x_{n+1}$ for all $n \in \mathbb{N}$ and x_n is an increasing sequence.

Standard homework problem on induction.
5 marks

(iii) *Base case*

$$|x_1 - x_0| = \left|\frac{5}{4} - 1\right| = \frac{1}{4}.$$

So the required upper bound on $|x_{n+1} - x_n|$ holds for $n = 0$.
Inductive step Now suppose that the required upper bound holds on $|x_{n+1} - x_n|$. Then we use the formula for $|x_{n+2} - x_{n+1}|$ at the start of (ii). We also use the bounds $x_n \geq 1$ and $x_{n+1} \geq 1$ to deduce

$$(3 + x_n)(3 + x_{n+1}) \geq 4 \times 4 = 16.$$

Then from (ii) we have

$$|x_{n+2} - x_{n+1}| = \frac{7|x_{n+1} - x_n|}{(3+x_n)(3+x_{n+1})} \leq \frac{7}{16}|x_{n+1} - x_n|$$

$$\leq \frac{7}{16} \cdot \left(\frac{7}{16}\right)^n \cdot \frac{1}{4} = \left(\frac{7}{16}\right)^{n+1} \cdot \frac{1}{4}.$$

So the upper bound for $|x_{n+1} - x_n|$ implies the upper bound for $|x_{n+2} - x_{n+1}|$, and by induction we have

$$|x_{n+1} - x_n| \leq \frac{1}{4}\left(\frac{7}{16}\right)^n$$

for all $n \in \mathbb{N}$.

15 marks in total

Theory from lectures 5 marks	11. A set $A \subset Q$ is a *Dedekind cut* if (i) A is nonempty, and bounded above, (ii) $x \in A \wedge y \in \mathbb{Q} \wedge y < x \Rightarrow y \in A$ (iii) A does not have a maximal element.
Similar to homework exercises	
1 mark	a) $A = \{x \in \mathbb{Q} : x \leq 6.5\}$ has a maximal element (6.5)¿ So property (iii) is violated and A is not a Dedekind cut.
1 mark	(b) $A = \{x \in \mathbb{Q} : 7 < x\}$ is not bounded above – because, for example, A contains all integers ≥ 8. So property (i) is violated and A is not a Dedekind cut.
3 marks	c) $$A = \left\{x : (x - \frac{3}{2})^2 < \frac{5}{4}\right\} = \left\{x : \frac{3}{2} - \frac{\sqrt{5}}{2} < x < \frac{3}{2} + \frac{\sqrt{5}}{2}\right\}.$$ So $0 \notin A$ but $\frac{3}{2} \in A$ (for example). So property (ii) is violated, and A is not a Dedekind cut.
Special case of theory from lectures 4 marks	We check the properties of $2A$ one by one. (i) $x \in 2A \wedge y < x \Leftrightarrow \frac{x}{2} \in A \wedge \frac{y}{2} < \frac{x}{2} \Rightarrow \frac{y}{2} \in A \Rightarrow y \in 2A$. (ii) $A \neq \emptyset \Rightarrow \exists x \in A \Rightarrow \exists 2x \in \ 2A \Rightarrow 2A \neq \emptyset$. (iii) $\exists M, x \leq M \forall x \in A \Rightarrow y \leq 2M \forall y \in A$. So $2A$ is bounded above. $$\exists b \in 2A, y \leq b \forall y \in 2A \Rightarrow \frac{b}{2} \in A \wedge x \leq \frac{b}{2} \forall x \in A$$ So as A does not have a maximal element, $2A$ does not either. So 2A is a Dedekind cut.
Theory from lectures, but only incidentally, so unseen. 1 mark 15 marks in total	By the second property of a Dedekind cut, if $x \in A$ then $y \in A$ for all $y \in \mathbb{Q}$ with $y < x$ and hence $z \in -A$ for all $z \in \mathbb{Q}$ with $z > -x$. So $-A$ is not bounded above and is not a Dedekind cut.

Theory from lectures 4 marks	12. A is *finite* if either A is empty, or, for some $n \in \mathbb{Z}_+$, there is a bijection $f : \{k \in \mathbb{N} : k < n\} \to A$. For a fixed set A, there is at most one $n \in \mathbb{Z}_+$ for which such a bijection exists, and if there is such an n then A is said to be *of cardinality* n. The empty set is said to be of cardinality 0. A is *countable* if either A is finite or there is a bijection $f : \mathbb{N} \to A$.
Standard examples 2 marks	a) $[0, 1]$ is uncountable, and $g : A \to B$ is injective, where $g(x) = \frac{x}{2}$ for all $x \in [0, 1]$
1 mark	b) $[0, 1)$ is uncountable, and $h : B \to A$ is injective where $h(x) = x$ for all $x \in [0, 1)$.
Theory from lectures Standard examples 1 mark	c) $\mathbb{Z}$ is countable.
1 mark	d) $\mathbb{N}^2$ is countable.
2 marks	The Schroder-Bernstein Theorem, says that if there exists an injective map $g : A \to B$ and an injective map $h : B \to A$ then there is a bijection $k : A \to B$. A composition of bijections is a bijection, so if one of A and B is in bijection with $\mathbb{N}$, the other one is too.
Theory from lectures, but should be regarded as unseen. No memorising required or desired. 4 marks 15 marks in total	The set $A_p = \{(m, n) \in \mathbb{N}^2 : m+n = p\}$ can be written as $\{(m, p-m) : 0 \leq p \leq m\}$, for all $p \in \mathbb{N}$, and so has $p + 1$ elements, and is therefore finite. Clearly we can write $\mathbb{N}^2 = \cup_{p=0}^{\infty} A_p$ and therefore $\mathbb{N}^2$ is a countable union of finite sets.