Solutions to MATH105 exam January 2011
Section A

3 marks	1.a) For a real number x, $x^2 + x - 2 = 0$ if and only if $x = 1$ or $x = -2$. This is true because $x^2 + x - 2 = (x+2)(x-1) = 0 \Leftrightarrow x + 2 = 0$ or $x - 1 = 0$.
3 marks	b) For a real number x, if x is greater than 0 and less than 3, then x is greater than 1 and less than 2. This is clearly false. For example if $x = 1$ then $1 > 0$ and $1 < 3$ but it is not true that $1 > 1$.
Standard homework exercises 6 marks in total	
1 mark	2a)$\exists x \in \mathbb{R}, x^2 + x + 1 \leq 0$
3 marks	b) $\exists x \in \mathbb{R}$, $\exists y \in \mathbb{R}$, $x > y \wedge x^2 \leq y^2$.
Standard homework exercises 4 marks in total	
1 mark	3a) Yes, $3 \in \pi$ because $\pi > 3$
1 mark	b) No $3 \notin X$ because $3 > 2$ and so $3 \notin [1, 2]$
1 mark	c) No $3 \notin X$.
1 mark	d) Yes
1 mark	e) No because $2 + 3i$ is not a real number
1 mark	f) No because π is not an integer – in fact not even rational.
Standard homework exercises 6 marks in total	
1 mark	4a) $1 - 3x > 5 \Leftrightarrow 3x < -4 \Leftrightarrow x < -4/3$.
2 marks	b) If $1 - x > 0$ then $2 < \frac{1+x}{1-x} < 3 \Leftrightarrow 2 - 2x < 1 + x < 3 - 3x \Leftrightarrow$ $(1 < 3x \wedge 4x < 2 \Leftrightarrow \frac{1}{3} < x < \frac{1}{2}$, which is compatible with $x < 1$.
2 marks	If $1 - x < 0$ then $2 < \frac{1+x}{1-x} < 3 \Leftrightarrow 2 - 2x > 1 + x > 3 - 3x \Leftrightarrow$ $1 > 3x \wedge 4x > 2 \Leftrightarrow \frac{1}{3} > x \wedge x > \frac{1}{2}$. This is never true. So altogether we have $2 < \frac{1+x}{1-x} < 3 \Leftrightarrow \frac{1}{3} < x < \frac{1}{2}$ Alternatively, it would be permissible to sketch the graph.
Standard homework exercises. 5 marks in total	

1 marks

5. To start the induction, $10^3 = 1000 < 1024 = 2^{10}$. So $n^3 < 2^n$ is true for $n = 10$.
Now suppose inductively that $n \geq 10$ and $n^3 < 2^n$. Then

5 marks

$$(n+1)^3 = n^3+3n^2+3n+1 < n^3+3n^2+3n^2+n^2 = n^3+7n^2 < 2n^3 < 2\cdot 2^n = 2^{n+1}$$

So true for n implies true for $n+1$ and $n^3 < 2^n$ is true for all $n \geq 10$.

Standard homework exercise
6 marks in total

6.

$$\left(\begin{array}{cc|c} 1 & 0 & 351 \\ 0 & 1 & 279 \end{array}\right) \xrightarrow{R_1 - R_2} \left(\begin{array}{cc|c} 1 & -1 & 72 \\ 0 & 1 & 279 \end{array}\right) \underset{R_2 - 3R_1}{\rightarrow} \left(\begin{array}{cc|c} 1 & -1 & 72 \\ -3 & 4 & 63 \end{array}\right)$$

$$\xrightarrow{R_1 - R_2} \left(\begin{array}{cc|c} 4 & -5 & 9 \\ -3 & 4 & 63 \end{array}\right) \underset{R_2 - 7R_1}{\rightarrow} \left(\begin{array}{cc|c} 4 & -5 & 9 \\ -31 & 39 & 0 \end{array}\right)$$

4 marks

As a result of this:

1 mark — (i) the g.c.d. d is 9;

1 mark — (ii) from the last row of the last matrix, $r = 39$ and $s = 31$;

1 mark — (iii) from the first row of either of the last two matrices $m = 4$ and $n = -5$;

2 marks — (iv) The lcm is $351 \times 31 = 10881$.

Standard homework exercise
9 marks in total

3 marks

7a) $f((-1,\infty)) = [0,\infty)$ because $x^2 \geq 0$ for all $x \in \mathbb{R}$ and $f(\sqrt{y}) = y$ for all $y \geq 0$. So the image of f is $[0,\infty)$ and f is not surjective. Also, f is not injective, because $f(x) = f(-x)$ for all $x \in (0,1)$, for which $-x \in (-1,0) \subset (-1,\infty)$.

4 marks

b) $f(x) = y \Leftrightarrow y = \dfrac{x}{x+1} \Leftrightarrow xy + y = x \Leftrightarrow x(y-1) = y \Leftrightarrow x = \dfrac{y}{y-1}$. Now $\dfrac{y}{y-1}$ is defined for $y \in \mathbb{R} \Leftrightarrow y \neq 1$. So the image of f is $(-\infty,1) \cup (1,\infty) \neq \mathbb{R}$ and f is not surjective. However, f is injective, because, for any $y \neq 1$, the only value of x for which $f(x) = y$ is $x = \dfrac{y}{y-1}$.

Standard homework exercise
7 marks in total

3 marks	8a) Since the image of the map $f(x) = x^2 + 1$ is the set $[1, \infty)$, a conditional definition of this set is $\{x \in \mathbb{R} : x \geq 1\}$
3 marks	The integers ≥ 2 with 3 as the only prime factor are precisely the numbers of the form 3^m for $m \in \mathbb{Z}_+$. So a constructive definition of this set is $\{3^m : m \in \mathbb{Z}_+\}$.
Standard homework exercise 6 marks in total	
1 mark	9a)This is an increasing sequence since $n^3 < (n+1)^3$ for all integers $n \geq 1$.
3 marks	b) $n^2 - 7n + 10 = (n-2)(n-5)$. So $x_1 = 4$, $x_2 = x_5 = 0$ and $x_3 = x_4 = -2$. So $x_1 > x_2$ but $x_4 < x_5$ (for example). So this sequence is neither increasing or decreasing.
2 marks	c) Since $x_n = 1 - \dfrac{2}{n^2+1}$ and $\dfrac{2}{n^2+1}$ is decreasing with n, we see that x_n is an increasing sequence.
Standard homework exercise 6 marks in total	

Section B

Theory from lectures
4 marks

10. $\sim$ is *reflexive* if

$$x \sim x \forall x \in X$$

$\sim$ is *symmetric* if

$$x \sim y \ \Rightarrow \ y \sim x \ \forall \ x, \ y \in X.$$

$\sim$ is *transitive* if

$$(x \sim y \wedge y \sim z) \ \Rightarrow \ x \sim z \ \forall \ x, \ y, \ \in X.$$

Standard homework exercise
2 marks

a) $\sim$ is reflexive because $x - x = 0 \in \mathbb{N}$ for all $x \in \mathbb{N}$. It is not symmetric because if $x, \ y \in \mathbb{N}$ then $x - y \in \mathbb{N} \Leftrightarrow x - y \geq 0$ So for example $2 \sim 1$ but it is not true that $1 \sim 2$. So $\sim$ is not an equivalence relation.

Standard homework exercise
4 marks

b) $x + x = 2x$ is even for all $x \in \mathbb{N}$ and so $\sim$ is symmetric
If $x, \ y \in \mathbb{N}$ and $x + y$ is even then $y + x = x + y$ is even and so $\sim$ is symmetric.
If $x, \ y, \ z \in \mathbb{N}$ and $x+y$ and $y+z$ are both even then $(x+y)+(y+z) = x + z + 2y$ is even. But then since $2y$ is even, $x + z$ is even. So $\sim$ is transitive and $\sim$ is an equivalence relation.

Standard homework exercise
1 mark

c) $1 + 1 = 2$ is not divisible by 3. So it is not true that $1 \sim 1$ and $\sim$ is not reflexive. So $\sim$ is not an equivalence relation.

Unseen: matrices are studied in the core module MATH103.
4 marks

d) If I denotes the 2×2 identity matrix then I is invertible with $I = I^{-1}$ and $A = IAI^{-1}$ for any 2×2 matrix A. So $\sim$ is reflexive.
If P is an invertible 2×2 matrix and $B = PAP^{-1}$, then P^{-1} is invertible and $(P^{-1})^{-1} = P$ and $A = P^{-1}BP$. So $\sim$ is symmetric.
If P and Q are invertible 2×2 matrices and $B = PAP^{-1}$ and $C = QBQ^{-1}$, then QP is invertible with inverse $P^{-1}Q^{-1}$ and $C = QBQ^{-1} = (QP)A(QP)^{-1}$. So $\sim$is transitive and $\sim$ is an equivalence relation.

15 marks in total

6 marks

11a) $1 = 1$ so the formula is true for $n = 0$. Now suppose inductively that

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1}.$$

Then

$$\sum_{k=0}^{n+1} x^k = \frac{x^{n+1} - 1}{x - 1} + x^{n+1} = \frac{x^{n+1} - 1 + x^{n+2} - x^{n+1}}{x - 1} = \frac{x^{n+2} - 1}{x - 1}.$$

So if the formula holds for n it also holds for $n+1$. So by induction it holds for all $n \geq 0$.

9 marks

b) When $n = 1$ the left-hand side of the formula is x and the right-hand side is

$$\frac{x^3 - 2x^2 + x}{(x-1)^2} = \frac{x(x-1)^2}{(x-1)^2} = x$$

So the formula is true for $n = 1$. Now suppose inductively that for some $n \geq 1$,

$$\sum_{k=1}^{n} kx^k = \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$$

Then

$$\sum_{k=1}^{n+1} x^k = \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2} + (n+1)x^{n+1}$$

$$= \frac{nx^{n+2} - (n+1)x^{n+1} + x + (n+1)x^{n+3} - 2(n+1)x^{n+2} + (n+1)x^{n+1}}{(x-1)^2}$$

$$= \frac{(n+1)x^{n+3} - (n+2)x^{n+2} + x}{(x-1)^2}$$

So if the formula is true for n is is true for $n+1$. So by induction it holds for all $n \geq 1$

Standard homework exercises
15 marks in total

Theory from lectures 3 marks

12(i) $|S \cup M \cup D| = |S| + |M| + |D| - |S \cap M| - |M \cap D| - |S \cap D| + |S \cap M \cap D|$.

Unseen but similar to homework exercises
4 marks

(ii)The number of people having at least two courses is

$$|(S \cap M) \cup (M \cap D) \cup (S \cap D)|.$$

The intersection of any two of the sets $S \cap M$, $M \cap D$, $S \cap D$ is $S \cap M \cap D$.So the intersection of all three of these sets is also $S \cap M \cap D$. Applying the inclusion-exclusion principle we have

$$|(S \cap M) \cup (M \cap D) \cup (S \cap D)|$$

$$= |S \cap M| + |M \cap D| + |S \cap D - 3|S \cap M \cap D| + |S \cap M \cap D|$$
$$= |S \cap M| + |M \cap D| + |S \cap D| - 2|S \cap M \cap D|.$$

Unseen but similar to homework exercises
4 marks

(iii) Adding the equations from (i) and (ii) the terms $|S \cap M| + |M \cap D| + |S \cap D|$ cancels and we obtain

$$|S \cup M \cup D| + |(S \cap M) \cup (M \cap D) \cup (S \cap D)| = |S| + |M| + |D| - |S \cap M \cap D|$$

Since 37 people have at least one course and 33 people have at least two courses,

$$70 = 27 + 36 + 21 - |S \cap M \cap D| = 84 - |S \cap M \cap D|$$

Thus, the number $|S \cap M \cap D|$ of people having all three courses is 14.

Similar to homework exercises

4 marks

(iv)$|M \cap (S \cup D)| = 36 - 4 = 32$. Applying the inclusion-exclusion principle to the two sets $M \cap S$ and $M \cap D$ we have

$$32 = |M \cap S| + |M \cap D| - |M \cap S \cap D| = 26 + |M \cap D| - 14.$$

So the number of people having the main course and the dessert is

$$|M \cap D| = 32 - 12 = 20.$$

Similar to homework exercises
15 marks in total

Theory from lectures 6 marks	13(i) A set $A \subset Q$ is a *Dedekind cut* if • A is nonempty, and bounded above, • $x \in A \wedge y < x \Rightarrow y \in A$ • A does not have a maximal element.
Similar to homework exercises 1 mark 2 marks	(ii)a) Q is not bounded above, so not a Dedekind cut (ii)b) $0 \in \mathbb{Q}$ but $-1 \notin \mathbb{Q}$ and $-1 < 0$ so A is not a Dedekind cut
6 marks	(iii)A is bounded above – by 2 for example because if $x \geq 2$ then $x^3 \geq 8 > 2$. and since $x \mapsto x^3$ is strictly increasing, if $a \in A$ and $b < a$ then $b^3 < a^3 < 2$, so $b \in A$. Finally, we see that A has no maximal element as follows. Let $a \in A$ with $1 \leq a$ and let $0 < \varepsilon < 1$. Then $a^2 \geq a \geq 1$ and $\varepsilon^3 < \varepsilon^2 < \varepsilon$. Then $(a+\varepsilon)^3 = a^3 + 3a^2\varepsilon + 3a\varepsilon + \varepsilon^3 < a^3 + 3a^2\varepsilon + 3a^2\varepsilon + a^2\varepsilon = a^3 + 7a^2\varepsilon.$ If we choose $\varepsilon \in \mathbb{Q}$ with $0 < \varepsilon < \dfrac{2-a^3}{7a^2}$ then $(a+\varepsilon)^3 < 2$ and $a + \varepsilon \in A$. Hence a is not maximal and A has no maximal element. So A is a Dedekind cut.
15 marks in total Standard homework exercise	