Solutions to MATH105 exam August 2014
Section A

1 mark 2 marks 2 marks 2 marks Standard homework exercises 7 marks in total. No reasons required.	1.a) $1 > 2$ and $3 < 6$ This is false. b) It is not the case that either $1 \leq 2$ or $4 \leq 2$. This is false (because it is the case). c) If x is real and $x^3 > 0$ then $x > 0$. This is true. d) For all real numbers x, $x^2 + x > 2$. This is false. In fact $x^2 + x \leq 2 \Leftrightarrow -2 \leq x \leq 1$.
1 mark 1 mark 2 marks 2 marks Standard homework exercises. 6 marks in total.	2a) $1 \leq 2 \vee 3 \geq 6$. b) $1 \leq 2 \vee 4 \leq 2$. c) $\exists x \in \mathbb{R}$, $x^3 > 0 \wedge x \leq 0$. d) $\exists x \in \mathbb{R}$, $x^2 + x \leq 2$.
1 mark 3 marks 1 mark Standard homework exercise. 5 marks in total.	3. **Base case**: When $n = 0$, $2^{0+1} + 3 = 5$, so $x_n = 2^{n+1} + 3$ is true when $n = 0$. **Inductive step**: Suppose that $n \geq 0$ and $x_n = 2^{n+1} + 3$. Then $$x_{n+1} = 2x_n - 3 = 2(2^{n+1} + 3) - 3 = 2^{n+2} + 6 - 3 = 2^{(n+1)+1} + 3.$$ So if $n \geq 0$, $x_n = 2^{n+1} + 3 \Rightarrow x_{n+1} = 2^{(n+1)+1} + 3$. So, by induction, $x_n = 2^{n+1} + 3$ for all integers $n \geq 0$.
2 marks 3 marks Standard homework exercise 5 marks in total	4. $1584 = 8 \times 198 = 16 \times 99 = 2^4 \times 3^2 \times 11$. The divisors are $2^{n_1} \times 3^{n_2} \times 11^{n_3}$, where n_1, n_2 and n_3 are integers with $0 \leq n_1 \leq 4$, $0 \leq n_2 \leq 2$ and $0 \leq n_3 \leq 1$, with each triple (n_1, n_2, n_3) giving a different divisor. So the number of divisors is $5 \times 3 \times 2 = 30$.

1 mark	5a) $((0,2] \cap [1,3]) \cap [0,4] = [1,2] \cap [0,4] = [1,2]$.
1 mark	b) $(0,2] \cap [1,5]) \cup [2,4] = [1,2] \cup [2,4] = [1,4]$.
1 mark	c) $((0,2] \cup [1,3]) \setminus [2,4] = (0,3] \setminus [2,4] = (0,2)$.
Standard homework exercises. 3 marks in total.	

6.

$$\begin{array}{cc|c} 1 & 0 & 352 \\ 0 & 1 & 213 \end{array} \xrightarrow{R_1 - R_2} \begin{array}{cc|c} 1 & -1 & 139 \\ 0 & 1 & 213 \end{array} \xrightarrow[R_2 - R_1]{} \begin{array}{cc|c} 1 & -1 & 139 \\ -1 & 2 & 74 \end{array}$$

$$\xrightarrow{R_1 - R_2} \begin{array}{cc|c} 2 & -3 & 65 \\ -1 & 2 & 74 \end{array} \xrightarrow[R_2 - R_1]{} \begin{array}{cc|c} 2 & -3 & 65 \\ -3 & 5 & 9 \end{array}$$

$$\xrightarrow{R_1 - 7R_2} \begin{array}{cc|c} 23 & -38 & 2 \\ -3 & 5 & 9 \end{array} \xrightarrow[R_1 - 4R_2]{} \begin{array}{cc|c} 23 & -38 & 2 \\ -95 & 157 & 1 \end{array}$$

4 marks	
	As a result of this:
1 mark	(i) the g.c.d. d is 1;
1 mark	(ii) Since the gcd is 1, $m_1 = 213$ and $n_1 = 352$;
1 mark	(iii) from the second row of either of the last two matrix $a = 157$ and $b = -95$;
2 marks	(iv) The l.c.m. is $213 \times 352 = 74976$.
Standard homework exercise. 9 marks in total	

1 mark	7. $f : X \to Y$ is *strictly increasing* if, whenever x_1, $x_2 \in X$ with $x_1 < x_2$, we have $f(x_1) < f(x_2)$.
1 mark	$f : X \to Y$ is *injective*, if, whenever x_1, $x_2 \in X$ with $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$.
2 marks	Suppose that $f : X \to Y$ is strictly increasing, and suppose that x_1, $x_2 \in X$ with $x_1 \neq x_2$. THen either $x_1 < x_2$ or $x_2 < x_1$. After renaming the points if necessary, we can assume that $x_1 < x_2$. Then since f is strictly increasing, we have $f(x_1) < f(x_2)$, and hence $f(x_1) \neq f(x_2)$. Since $\{x_1, x_2\}$ is an arbitrary set of two points in X, it follows that f is injective .
1 mark	a) f is strictly increasing on the domain $[0, \infty)$, and hence is injective
1mark	b) $f(0) = f(2)$. So f is not injective.
2 marks	c) $1/x_1 = 1/x_2 \Leftrightarrow x_2 = x_1$ (multiplying the original equation through by x_2x_1). So f is injective.
Bookwork followed by two standard homework exercise and another bookwork exercise which was set in homework. 8 marks in total.	
2 marks	8. $\lvert A \cup B\rvert = \lvert A\rvert + \lvert B\rvert - \lvert A \cap B\rvert$
2 marks	Let A be the set of students studying Mathematics and let B be the set of students studying Finance. We are given that $$\lvert A \cup B\rvert = 127, \quad \lvert A\rvert = 105, \quad \lvert B\rvert = 56$$ So $$\lvert A \cap B\rvert = \lvert A\rvert + \lvert B\rvert - \lvert A \cup B\rvert = 105 + 56 - 127 = 161 - 127 = 34.$$
Book work followed by standard homework exercise. 4 marks in total.	

2 marks 2 marks 2 marks Bookwork followed by standard homework exercises. 6 marks	9. A real number x is *algebraic* if there are $n \in N$ and integers a_i, for $0 \leq i \leq n$, such that $\sum_{i=0}^{n} a_i x^i = 0$. a) If $x = 2 + \sqrt{2}$, then $x - 2 = \sqrt{2}$ and $(x-2)^2 = 2$, that is, $x^2 - 4x + 4 = 2$, and $x^2 - 4x + 2 = 0$, and x is algebraic. b) If $y = \sqrt{2 + \sqrt{2}}$, then $y^2 = x$, for x as in a). So $y^4 - 4y^2 + 2 = 0$, and y is algebraic.
1 mark 1 mark 1 mark Standard homework exercises. 3 marks	10. a) Countable. b) Uncountable. c) Countable.

Section B

Theory from lectures 3 marks	11. $\sim$ is *reflexive* if $$x \sim x, \quad \forall x \in X$$ $\sim$ is *symmetric* if $$x \sim y \;\Rightarrow\; y \sim x, \quad \forall\, x,\; y \in X.$$ $\sim$ is *transitive* if $$(x \sim y \wedge y \sim z) \;\Rightarrow\; x \sim z, \quad \forall\, x,\; y,\; \in X.$$
Theory from lectures. 2 marks	The equivalence class $[x]$ of x is the set $\{y \in X : y \sim x\}$.
Standard homework exercise 1 mark	a) $n \geq n$ for all integers n. So $\sim$ is *reflexive*. However $2 \geq 1$ and $\neg$ (1 ≥ 2 so $\sim$ is not symmetric and not an equivalence relation.
Standard homework exercise. 3 marks	b) For any $x \in \mathbb{R}$, $x - x = 0 \in \mathbb{Z}$. So $\sim$ is reflexive. If $x \sim y$, the $x - y \in \mathbb{Z}$ and hence $y - x = -(x - y) \in \mathbb{Z}$ and $y \sim x$, so $\sim$ is symmetric. If $x \sim y$ and $y \sim z$ then $x - y \in \mathbb{Z}$ and $y - z \in \mathbb{Z}$ and hence $x - z = (x - y) + (y - z) \in \mathbb{Z}$ and $x \sim z$. So $\sim$ is transitive and $\sim$ is an equivalence relation.
Standard exercise, with notation likely to prove more challenging 4 marks	c)$x/x = 1 = 2^0$ for any $x \in \mathbb{Q} \setminus \{0\}$. So $x \sim x$ for any $x \in \mathbb{Q} \setminus \{0\}$ and $\sim$ is *reflexive*. If $x,\, y \in \mathbb{Q} \setminus \{0\}$ and $x \sim y$, then $x/y = 2^n$ for some $n \in \mathbb{Z}$ and $y/x = 2^{-n}$. Since $-n \in \mathbb{Z}$ we have $y \sim x$. Since x and y can be interchanged, we have $x \sim y \Leftrightarrow y \sim x$, and $\sim$ is *symmetric*. If $x,\, y,\, z \in \mathbb{Q}\setminus\{0\}$ and $x \sim y$ and $y \sim z$, then $x/y = 2^{n_1}$ and $y/z = 2^{n_2}$ for some $n_1,\, n_2 \in \mathbb{Z}$, and $x/z = x/y \times y/z = 2^{n_1+n_2}$, and since $n_1 + n_2 \in \mathbb{Z}$, we have $x \sim z$ So $(x \sim y \wedge y \sim z) \Rightarrow x \sim z$ and $\sim$ is *transitive*. So $\sim$ is an equivalence relation.
Unseen 2 marks	The equivalence classes of the (positive) primes are all disjoint. For suppose p_1 and p_2 are distinct primes. If $p_1 = p_2 \times 2^n$ for $n \in \mathbb{Z}_+$, then we have two ways of writing p_1 as a product of powers of distinct primes (even if $p_2 = 2$), giving a contradiction. If $p_1 = p_2 \times 2^{-n}$ for $n \in \mathbb{Z}$ then we have $p_2 = p_1 \times 2^n$, giving two ways of writing p_2 as a product of powers of distinct primes, which again gives a contradiction. So $[p_1] \neq [p_2]$ whenever p_1 and p_2 are distinct primes. Since there are infinitely many primes, there are infinitely many equivalence classes
15 marks in total.	

Standard homework exercise 1 mark	12a) *Base case*: $3 \times 8^2 + 5 = 197 < 256 = 2^8$, so $3n^2 + 5 < 2^n$ is true for $n = 8$.
4 marks	*Inductive step*: Suppose that for some $n \in \mathbb{Z}$ with $n \geq 8$ we have $3n^2 + 5 < 2^n$. Then $$3(n+1)^2 + 5 = 3n^2\left(1+\frac{1}{n}\right)^2 + 5 \leq \frac{81}{64} \times 3n^2 + 5 < \frac{81}{64}(3n^2+5)$$ $$< \frac{81}{64} \times 2^n < 2 \times 2^n = 2^{n+1}$$
1 mark	So if the inequality holds for some integer $n \geq 8$, it also holds for $n+1$. So by induction $3n^2 + 5 < 2^n$ holds for all integers $n \geq 8$.
Unseen: extra exercise on problem sheet about using induction to prove the associative law for addition of natural numbers. 1 mark	b) *Base case*: The base case $n = 1$ is simply $1 + 1 = 1 + 1$, and it is clear that this is true.
2 marks	*Inductive step*: Fix $n \in \mathbb{Z}_+$ and suppose that $n + 1 = 1 + n$. Then $(n+1)+1 = (1+n)+1$. We are allowed to assume that $(1+n)+1 = 1+(n+1)$. So we have $(n+1)+1 = 1+(n+1)$.
1 mark	So is $n + 1 = 1 + n$ we also have $(n+1)+1 = 1+(n+1)$. So by induction, $n + 1 = 1 + n$ for all $n \in \mathbb{Z}_+$
1 mark	*Base case* So now the base case of $n + m = m + n$ holds for $m = 1$ and for all $n \in \mathbb{Z}_+$.
3 marks	*Inductive step* Now for a fixed n, $m \in \mathbb{Z}_+$, suppose that $n + m = m + n$.Then $$n + (m+1) = (n+m)+1 = (m+n)+1 = m+(n+1)$$ $$= m + (1+n) = (m+1)+n.$$
1 mark	So if $n + m = m + n$ we also have $n + (m+1) = (m+1) + n$. So by induction on m, $n + m = m + n$ for all $m \in \mathbb{Z}_+$ (for any $n \in \mathbb{Z}_+$).
15 marks in total	

Bookwork 4 marks	13 $f : X \to Y$ is *injective* if, whenever x_1, $x_2 \in X$ and $f(x_1) = f(x_2)$, then $x_1 = x_2$. $f : X \to Y$ is *surjective* if $\mathrm{Im}(f) = Y$, where $\mathrm{Im}(f) = \{f(x) : x \in X\}$. $f : X \to Y$ is a *bijection* if $f : X \to Y$ is injective and surjective.
Bookwork, and similar exercise 4 marks	Suppose that $f : X \to Y$ and $g : Y \to Z$ are both injective. Suppose that x_1 and $x_2 \in X$ and $g \circ f(x_1) = g \circ f(x_2)$, that is, $g(f(x_1)) = g(f(x_2))$. Then since g is injective we have $f(x_1) = f(x_2)$, and since f is injective, we have $x_1 = x_2$. So $g \circ f$ is injective.
Bookwork 2 marks	A is *countable* if either it is empty, or there is an injective map $f : A \to \mathbb{Z}$. ($\mathbb{Z}_+$ or $\mathbb{N}$ can also be used as the codomain.)
Set in homework 5 marks	We are allowed to assume the base case $n = 2$. Also, it is clear that $\mathbb{Z} = \mathbb{Z}^1$ is countable. Suppose that $n \geq 2$ and $\mathbb{Z}^n$ is countable. Therefore there is an injective map $f_n : \mathbb{Z}^n \to \mathbb{Z}$ (which is also a bijection, but we do not need this). The map $F : \mathbb{Z}^{n+1} \to \mathbb{Z} \times \mathbb{Z}^n$ given by $F(m_1, m_2, \cdots m_{n+1}) = (m_1, (m_2, \cdots m_{n+1}))$ is a bijection. The map $G : \mathbb{Z} \times \mathbb{Z}^n \to \mathbb{Z}^2$ given by $G(m_1, (m_2, \cdots m_{n+1})) = (m_1, f_n(m_2, \cdots m_{n+1}))$ is also injective. So $G \circ F$ is injective, and $f_2 \circ G \circ F : \mathbb{Z}^{n+1} \to Z$ is injective. So if $\mathbb{Z}^n$ is countable, $\mathbb{Z}^{n+1}$ is also countable. So by induction, $\mathbb{Z}^n$ is countable for all $n \in \mathbb{Z}_+$.
15 marks in total.	
Theory from lectures 4 marks Similar to homework exercises.	14. A set $A \subset \mathbb{Q}$ is a *Dedekind cut* if a) $A \neq \emptyset$ b) $\mathbb{Q} \setminus A \neq \emptyset$ c) $x \in A \wedge y \in \mathbb{Q} \wedge y < x \Rightarrow y \in A$; d) A does not have a maximal element.
2 marks	(i) $x^2 + x + 3 = (x + \frac{1}{2})^2 + \frac{11}{4} > 0$ for all $x \in \mathbb{Q}$. So $A = \mathbb{Q}$ and A is not a Dedekind cut
4 marks	(ii) $0 \in A$ and $2 \notin A$, so properties a) and b) hold. If $f(x) = x^2 + x - 3$ then $-3 \notin A$ but $0 \in A$. So c) does not hold and A is not a Dedekind cut.
5 marks	(iii) $1 \in A$ and $2 \notin A$, so properties a) and b) hold. If $-1 \leq x \leq 1$ then $x^3 - x < 2$ and hence $f(x) = x^3 - x - 3 < 0$. Also, $f'(x) = 3x^2 - 1$ is > 0 of $x \leq -1$ or $x \geq 1$. So f is strictly increasing on each of the intervals $(-\infty, -1)$ and $(1, \infty)$. So if $x \leq -1$, $f(x) < 0$, and if $x \in A$ and $y < x$, then $y \in A$ if $y \leq 1$, and if $1 \leq y$ we have $f(y) < f(x) < 0$. So property c) holds. Finally, if $a \in A$ then by continuity of f we have $f(a + 1/n) < 0$ for a; sufficiently large $n \in \mathbb{Z}_+$. So a is not maximal for any $a \in A$. So A is a Dedekind cut.
15 marks in total	