Solutions to MATH105 exam September 2013
Section A

1 mark	1.a) -3 is an integer. This is true.
2 marks	b) There is a real number such that $x^2 + x < -1$. This is false (because $x^2 + x + 1 = (x + \frac{1}{2})^2 + \frac{3}{4} > 0$ for all $x \in \mathbb{R}$).
2 marks	c) For any integer n, 2 divides n if and only if 2 divides n^2. This is true.
2 marks	d) For all real numbers x and y, $x \leq y$ if and only if $x^2 \leq y^2$. This is false (because $-2 \leq 1$ but $4 \geq 1$, for example).
Standard homework exercises. No reasons required. 7 marks in total	
2 marks	2a)$x^2 \geq 9 \Leftrightarrow x \geq 3 \vee x \leq -3$. So $\{x : x^2 < 9\} = (-\infty, -3] \cup [3, \infty)$.
4 marks	b) For $\dfrac{x}{x-2} > 2$ either both x and $x + 2$ have to be > 0 or both < 0. If they are both positive we must have, in addition, $x > 2x - 4$, that is, $x < 4$. If they are both negative, we must have in addition that $x < 2x - 4$, that is, $x > 4$, which is impossible. So we must have $0 < x < 4$, that is, $\displaystyle \left\{x \in \mathbb{R} : \frac{x}{x+2} > 3\right\} = (0, 4).$
Standard homework exercises. 6 marks in total	
1 mark	3. **Base case**: When $n = 4$ both n^4 and 4^n are 4^4, so $n^4 \leq 4^n$ is true when $n = 4$.
4 marks	**Inductive step** Suppose that $n \geq 4$ and $n^4 \leq 4^n$. Then $\displaystyle (n+1)^4 = n^4 \times \left(1 + \frac{1}{n}\right)^4 \leq n^4 \times \left(\frac{5}{4}\right)^4 = \frac{625}{256} n^3 < 3n^4 \leq 3 \times 4^n < 4^{n+1}.$ So if $n \geq 4$, $n^4 \leq 4^n \Rightarrow (n+1)^4 \leq 4^{n+1}$.
1 mark	So, by induction, $n^4 \leq 4^n$ for all integers $n \geq 4$.
Standard homework exercises. 6 marks in total.	

6 marks Standard home-work exercises. 6 marks in total.	4. $1672 = 8 \times 209 = 2^3 \times 11 \times 19$. So the divisors of 1672 are $2^{k_1} \times 11^{k_2} \times 19^{k_3}$ for integers $0 \leq k_1 \leq 3$, $0 \leq k_2 \leq 1$, $0 \leq k_3 \leq 1$, that is, 1, 2, 4, 8, 11, 22, 44, 88, 19, 38, 76, 152, 209, 418, 836, 1672.
3 marks 2 marks Bookwork. 5 marks in total.	5. If m divides n then $n = mn_1$ for some $n_1 \in \mathbb{Z}$. If n divides p then $p = np_1$ for some $p_1 \in \mathbb{Z}$. If $m \mid n$ and $n \mid p$, we have $p = np_1 = m(n_1p_1)$ and since $n_1p_1 \in \mathbb{Z}$, it follows that $m \mid p$.
4 marks 1 mark 1 mark 1 mark 2 marks Standard home-work exercise. 9 marks in total	6. $\begin{array}{cc\|c} 1 & 0 & 748 \\ 0 & 1 & 231 \end{array} \overset{R_1 - 3R_2}{\longrightarrow} \begin{array}{cc\|c} 1 & -3 & 55 \\ 0 & 1 & 231 \end{array} \underset{R_2 - 4R_1}{\longrightarrow} \begin{array}{cc\|c} 1 & -3 & 55 \\ -4 & 13 & 11 \end{array}$ $\overset{R_1 - 5R_2}{\longrightarrow} \begin{array}{cc\|c} 21 & -68 & 0 \\ -4 & 13 & 11 \end{array}$ As a result of this: (i) the g.c.d. d is 11; (ii) from the first row of the last matrix, $m_1 = 68$ and $n_1 = 21$; (iii) from the second row of either of the last two matrices $a = -4$ and $b = 13$; (iv) The l.c.m. is $748 \times 21 = 231 \times 68 = 15708$.

2 marks 2 marks 4 marks Standard theory followed by standard homework exercise. 8 marks in total	7 $f : X \to Y$ is injective if $\forall x_1,\ x_2 \in X,\ f(x_1) = f(x_2 \Leftrightarrow x_1 = x_2$. The image of f, $\text{Im}(f)$ is $\{f(x) : x \in X\}$. f is a *bijection* if f is injective and $\text{Im}(f) = Y$, that is, f is also surjective $y = \dfrac{2x-1}{x+2} \Leftrightarrow 2x - 1 = y(x+2) = xy + 2y \Leftrightarrow x(2-y) = 2y + 1 \Leftrightarrow x = \dfrac{2y+1}{2-y}$. It follows that f is injective. Also, we see that $x > 0 \Leftrightarrow 0 < y < 2$. So $\text{Im}(f) = (0, 2)$.
2 marks 3 marks Standard homework exercises: unseen element in second one. 5 marks in total.	8. (i) $\dfrac{\sqrt{3}+2}{5}$ is rational $\Leftrightarrow$ $\dfrac{\sqrt{3}+2}{5} - \dfrac{2}{5} = \dfrac{\sqrt{3}}{5}$ is rational, $\Leftrightarrow$ $\sqrt{3}$ is rational. Since $\sqrt{3}$ is not rational, it follows that $\dfrac{\sqrt{3}+2}{5}$ is not either. (ii) Provided $x \neq -\dfrac{1}{3}$, $$x = \frac{3x+5}{3x+1} \Leftrightarrow x(3x+1) = 3x+5 \quad \Leftrightarrow \quad 3x^2 - 2x - 5 = 0$$ $$\Leftrightarrow \quad (3x-5)(x+1) = 0.$$ So the positive solution to this is $x = \dfrac{5}{3}$, which is rational.
1 mark 1 mark 1 mark Standard homework exercises 3 marks in total.	9(i) uncountable; (ii) countable; c) countable.

Section B

Theory from lectures
3 marks

10. $\sim$ is *reflexive* if

$$x \sim x \forall x \in X$$

$\sim$ is *symmetric* if

$$x \sim y \;\Rightarrow\; y \sim x \;\forall\; x,\; y \in X.$$

$\sim$ is *transitive* if

$$(x \sim y \wedge y \sim z) \;\Rightarrow\; x \sim z \;\forall\; x,\; y,\; \in X.$$

Theory from lectures
2 marks

The equivalence class $[x]$ of x is the set $\{y \in X : y \sim x\}$.

Standard homework exercise.
1 mark

(i) For this relation, $1 \sim 2$ and it is not true that $2 \sim 1$. So $\sim$ is not symmetric, and is not an equivalence relation.

Standard homework exercise.
3 marks

(ii) For any nonzero integer m, $m^2 > 0$ and so $\sim$ is *reflexive* on $\mathbb{Z} \setminus \{0\}$. For any integers m and n, $mn = nm$ and hence $mn > 0 \Leftrightarrow nm > 0$. So $\sim$ is *symmetric* If m, n and p are all non-zero integers and $m \sim n$ and $n \sim p$, and m, then m and n must both be positive or both negative, since $mn > 0$. Similarly n and p must both be positive or both negative. So m and p must both be positive or both negative, and hence $mp > 0$ and $m \sim p$. So $\sim$ is *transitive* and $\sim$ is an equivalence relation

Harder exercise, not previously set.
3 marks

(iii) For any $(x_1, y_1) \in \mathbb{R}$, $x_1 - x_1 + 2(y_1 - y_1) = 0 + 0 = 0$. So $\sim$ is reflexive.
For any (x_1, y_1) and $(x_2, y_2) \in \mathbb{R}$,

$$(x_2 - x_1) + 2(y_2 - y_1) = -((x_1 - x_2) + 2(y_1 - y_2)).$$

So $x_1 - x_2 + 2(y_1 - y_2) = 0 \Leftrightarrow x_2 - x_1 + 2(y_2 - y_1) = 0$, and $\sim$ is symmetric.
For any (x_1, y_1), (x_2, y_2), $(x_3, y_3) \in \mathbb{R}$

$$((x_1 - x_2) + 2(y_1 - y_2)) + (x_2 - x_3) + 2(y_2 - y_3)) = (x_1 - x_3) + 2(y_1 - y_3).$$

So

$$(x_1 - x_2 + 2(y_1 - y_2) = 0) \wedge (x_2 - x_3 + 2(y_2 - y_3) = 0) \;\Rightarrow\; (x_1 - x_3) + 2(y_1 - y_3) = 0,$$

and $\sim$ is transitive.

3 marks

Fix $(x_1, y_1) \in \mathbb{R}^2$ and write $c - x_1 + 2y_1$. Then $(x, y) \sim (x_1, y_1)$ $\Leftrightarrow x + 2y = c$, that is, if and only if (x, y) is on the straight line $x + 2y = c$. The equivalence class of $(a, 0)$ is the straight line $x + 2y = a$, or in parametric form $\{(a - 2t, t) : t \in \mathbb{R}\}$.

15 marks in total.

Standard homework exercise. 1 mark	10(i) . *Base case*: $1^2 = 1 = (1 \times (1+1) \times (2+1))/6$. So the formula holds for $n = 1$
5 marks	*Inductive step* Suppose that for some $n \in \mathbb{Z}_+$ we have $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. Then

$$\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2$$

$$= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = (n+1)\left(\frac{n(2n+1)}{6} + n + 1\right)$$

$$= \frac{(n+1)(n(2n+1)+6n+6}{6} = \frac{(n+1)(2n^2+7n+6)}{6}$$

$$\frac{(n+1)(2n+3)(n+2)}{6} = \frac{(n+1)(n+1+1)(2(n+1)+1}{6}.$$

So

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad \Rightarrow \quad \sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+1+1)(2(n+1)+1)}{6}.$$

1 mark	So by induction $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ holds for all $n \in \mathbb{Z}_+$.
Proved in lectures. 1 mark	(ii) *Base case* 0 is divisible by 2. So the statement is true for $n = 0$.
4 marks	*Inductive step* Suppose that exactly one of n, $n-1$ is divisible by 2. Since $n+1 = (n-1)+2$ is divisible by 2 if and only if $n-1$ is, exactly one of n and $n-1$ is divisible by 2 if and only if exactly one of $n = (n+1)-1$ and $n+1$ is divisible by 2. So if the statement is true for n, it is true for $n+1$.
1 mark	So by induction, for all $n \in \mathbb{N}$, exactly one of n and $n-1$ is divisible by 2.
3 marks	If $n \in \mathbb{N}$ is divisible by 2 then $n = 2k$ for some $k \in \mathbb{N}$. If $n \in \mathbb{N}$ and $n-1$ is divisible by 2 then $n-1 = 2k$ for some $k \in \mathbb{Z}_+ \subset N$ and $n = 2k+1$ for some $k \in \mathbb{N}$.
15 marks in total.	

Theory from lectures.
2 marks

11(i) The inclusion/exclusion principle for two finite sets A and B is that

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Standard homework exercises.
2 marks

(ii) If V is the set of people who had vanilla and B is the set of people who had another flavour then $|B| = 164$, $|V| = 139$ and $|V \cup B| = 215$. The set of people who bought had both vanilla and another flavour i is $V \cap B$ and from (i) we have

$$|V \cap B| = |V| + |B| - |V \cup B| = 139 + 164 - 215 = 303 - 215 = 88.$$

2 marks

(iii) the set of people who had only vanilla is $V \setminus B$. We have:

$$|V \setminus B| = |V| - |V \cap B| = 139 - 88 = 51.$$

1 mark

(iv) The set of people not having vanilla is $B \setminus V = (V \cup B) \setminus V$, so we have

$$|B \setminus V| = |V \cup B| - |V| = 215 - 139 = 76.$$

2 marks

(v) Let C be the set of people having chocolate. Since 31 of the 215 people had neither vanilla nor chocolate, 184 had either vanilla or chocolate, that is $|V \cup C| = 184$. We are also given $|V \cap C| = 28$. So from (i) with $A = V$ and $B = C$, we have

$$184 = |V \cup C| = |V| + |C| - |V \cap C| = |C| + 139 - 28 = |C| + 111,$$

that is, $|M| = 184 - 111 = 73$, so 73 people had chocolate icecream.

2 marks

(vi) The set of people who had chocolate and not vanilla is $C \setminus V$, and

$$|C \setminus V| = |C| - |V \cap C| = 73 - 28 = 45.$$

So 45 people had chocolate and not vanilla.

2 marks

(vii) In (i) we take $V \cap C$ and $V \cap O$ to replace A and B, because $V \cap (C \cup O) = (V \cap C) \cup (V \cap O)$. Also $(V \cap C) \cap (V \cap O) = V \cap C \cap O$. So this gives

$$|V \cap (C \cup O)| = |V \cap C| + |V \cap O| - |V \cap C \cap O|.$$

2 marks

(viii) Given that $|V \cap C| = 28$, and $|V \cap O| = 75$, and $|V \cap (C \cup O)| = 88$ from (ii), we have, from (vii)

$$88 = 28 + 75 - |V \cap C \cap O|$$

and hence $|V \cap C \cap O| = 103 - 88 = 15$, that is, 15 people had vanilla and chocolate and something else.

15 marks in total.

Theory from lectures. 4 marks	12. A set $A \subset \mathbb{Q}$ is a *Dedekind cut* if: a) A is nonempty, and bounded above; b) $x \in A \wedge y \in \mathbb{Q} \wedge y < x \Rightarrow y \in A$; c) A does not have a maximal element.
Similar to homework exercises.	
1 mark	(i) $A = \{x \in \mathbb{Q} : x > -1/5\}$ is not bounded above. So a) is violated and A is not a Dedekind cut.
2 marks	(ii) $1 \in A$ and $0 \notin A$ (for example) So property b) is violated and A is not a Dedekind cut.
4 marks	(iii) If $f(x) = x^3 - 12x + 20$, then $f'(x) = 3x^2 - 12$ has zeros at ± 12, and as $f''(x) = 6x$, we see that -2 is a local maximum and 2 is a local minimum, and f is strictly increasing on $(-\infty, -2]$ and on $[2, \infty)$, and strictly decreasing on $[-2, 2]$. Since $f(2) = 4 > 0$, we see that $f(x) \geq 4$ for all $x \geq -2$. So A is bounded above by -2. But $f(-5) < 0$, so $-5 \in A$ and A is nonempty. Since f is strictly increasing on $(-\infty, -2]$, if $f(x) < 0$ and $y < x$ then $f(y) < 0$, that is, property b) holds. If $x \in A$ is a maximal element in A then $f(x) < 0$. But then by continuity of f at x, there is a rational $\delta > 0$ such that for any $y \in \mathbb{Q}$ with $x \leq y \leq x + \delta$, we have $f(y) < 0$, that is $y \in A$, contradicting x being a maximal element, that is property c) holds for A. So A is a Dedekind cut.
4 marks	(iv) Once again, if $f(x) = x^3 - 12x + 1$, then $f'(x) = 3x^2 - 12$ has zeros at ± 2, -2 is a local maximum, and f is strictly increasing on $(-\infty, -2]$, with $f(-2) = 17 > 0$. By the definition of A, the set is bounded above by -2, and since $-2 \in A$, it is non-empty. As in (iii), property b) holds because f is strictly increasing on $(-\infty, -2]$ and, again as in (iii), property c) holds because of continuity of f. So A is a Dedekind cut.
15 marks in total.	