PAPER CODE NO.
MATH 105

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JANUARY 2014 EXAMINATIONS

Numbers and Sets

TIME ALLOWED: Two and a half hours.

INSTRUCTIONS TO CANDIDATES: Full marks will be obtained by complete answers to all questions in Section A and three questions in Section B. The best 3 answers in Section B will be taken into account.

SECTION A

1. Write down each of the following statements in ordinary English (apart from equations and inequalities), and state whether each one is true.

a) $2 < 3 \vee 5 < 4$.

b) $2 > 3 \wedge 4 \leq 5$.

c) $x \in \mathbb{Q} \Rightarrow x \in \mathbb{Z}$.

d) $\exists x \in \mathbb{R},\ x^2 + 2x + 1 < 0$.

[7 marks]

2. Write down the negatives of the statements in 1, without using the negation symbol $\neg$.

[6 marks]

3. Let x_n be defined for integers $n \geq 0$ by

$$x_0 = \frac{3}{2} \text{ and } x_{n+1} = \frac{2x_n}{3} + \frac{1}{3}.$$

Prove by induction that $1 < x_n < 2$ for all integers $n \geq 0$. [6 marks]

4. Define what it means for an integer m to divide an integer n. Prove that if m, n and p are integers, and m divides n and n divides p, then m divides p.

[6 marks]

5. Each of the following sets can be written as a single interval. Write down the interval in each case.

a) $(0, 2] \cup [1, 3] \cup [2, 4]$.

b) $((0, 2] \cap [1, 3]) \cup [2, 4]$.

c) $((0, 2] \cap [1, 3]) \setminus [2, 4]$.

[3 marks]

6. Let $m = 1014$ and $n = 455$. Let d be the greatest common divisor of m and n. Using the Euclidean algorithm or otherwise:

(i) compute d;

(ii) find integers m_1 and n_1 such that $m = dm_1$ and $n = dn_1$;

(iii) find integers a and b such that $d = ma + nb$;

(iv) find the l.c.m. of m and n.

[9 marks]

7.

Define the image of a function $f : X \to Y$. Find the image $\text{Im}(f)$ of $f : X \to Y$ where $f(x) = 2x + 1$ and:

a) $X = \mathbb{R}$ and $Y = \mathbb{R}$;

b) $X = [0, 1]$ and $Y = \mathbb{R}$.

Show that if $f : X \to Y$ and $g : Y \to Z$ are functions then $\text{Im}(g \circ f) \subset \text{Im}(g)$.

[8 marks]

8. Define what it means for a real number x to be algebraic. Show that the number $x = 1 + \sqrt{2}$ is algebraic. [4 marks]

9. Define what it means for a function $f : X \to Y$ to be *injective*. Define what it means for a set A to be countable. State without reasons which of the following sets is countable and which is uncountable.

a) $(-\infty, 0]$;

b) $\mathbb{Q}$;

c) A set A such that there exists an injective map $f : A \to \mathbb{Z}$.

[6 marks]

SECTION B

10.

(i) An equivalence relation $\sim$ on X (where $x \sim y$ means "x is equivalent to y") is *reflexive*, *symmetric* and *transitive*. Define what each of these three terms means. Also, give the definition of the equivalence class $[x]$ of an element x of X.

(ii) Determine for which of the following $\sim$ is an equivalence relation on X.

a) $X = \mathbb{Z}$ and $m \sim n$ if and only if m divides n.

b) $X = \mathbb{Z}$ and $m \sim n$ if and only if 5 divides $m - n$.

(iii) Now let X be the set of all polynomials with real coefficients, that is, of the form

$$f(x) = \sum_{i=0}^{n} a_i x^i$$

for $n \in \mathbb{N}$ and $a_i \in \mathbb{R}$, for $0 \leq i \leq n$. Show that $\sim$ is an equivalence relation, where $f(x) \sim g(x)$ if $f(0) = g(0)$. Also, show that the equivalence class of the polynomial $x^2 - x$ is all polynomials of the form $xg(x)$, where $g(x)$ varies over all polynomials with real coefficents.

[15 marks]

11.

a) Prove by induction that $n < 2^n - 1$ for all integers $n \geq 2$.

b) Let p_n denote the n'th smallest positive prime, so that $p_1 = 2$, $p_2 = 3$ and so on. Show that for all $n \in \mathbb{Z}_+$,

$$p_{n+1} \leq 1 + \prod_{i=1}^{n} p_i.$$

The proof of this is not by induction. Now prove by induction that for all $n \in \mathbb{Z}_+$,

$$\prod_{i=1}^{n} p_i < 2^{2^n} - 1.$$

[15 marks]

12. Define what it means for a function $f : X \to Y$ to be *injective* (also called one-to-one), and what it means for $f : X \to Y$ to be *surjective* (also called onto) and what it means for $f : X \to Y$ to be *a bijection*. Prove that if $f : X \to Y$ and $g : Y \to Z$ are both injective, then $g \circ f : X \to Z$ is also injective.

State the Schröder-Bernstein Theorem. Using this theorem or otherwise, show that there is a bijection between $(0, 1]$ and $[0, 1]$.

Assuming that there is a bijection from $\mathbb{Z}_+^2$ to $\mathbb{Z}_+$, show that if A_n is a countable set for each $n \in \mathbb{Z}_+$, then $\cup_{n=1}^{\infty} A_n$ is countable. You may assume that the sets A_n are disjoint, and all non-empty.

[15 marks]

13. Define what it means for $A \subset \mathbb{Q}$ to be a *Dedekind cut*.

Determine which (if any) of the following is a Dedekind cut, checking whether or not the properties needed for a set to be a Dedekind cut do hold.

a) $A = \{x \in \mathbb{Q} : x > 5\}$.

b) $A = \{x \in \mathbb{Q} : x < 5\}$.

c) $A = \{x : x \in \mathbb{Q},\ x^2 < 5 \vee x < 0\}$.

Hint: you may assume that if $2 \leq a \leq 3$ and $0 < \varepsilon < 1$ then $(a+\varepsilon)^2 < a^2 + 7\varepsilon$.

d) $A = \{x \in \mathbb{Q} : x^2 < 5 \vee x > 0\}$.

[15 marks]