MATH105: Solutions to Practice Problems 9

5. Suppose for contradiction that $x = \dfrac{p}{q}$ for $p,\ q \in \mathbb{Z}_+$. We can assume that p and q are the smallest possible positive integers for which this is true, and therefore that the g.c.d of p and q is 1. Then $px = q$ and $p^3x^3 = q^3$, that is $5p^3 = q^3$. Since 2 is prime we deduce that 2 is one of the prime factors of q and hence $q = 5q_1$ for some $q_1 \in \mathbb{Z}_+$. So $5p^3 = q^3 = 5^3q_1^3$ and $p^3 = 5^2q_1^3$. So then 5 must be a prime factor of p and 5 divides both p and q the g.c.d. of p and q is not 1, giving a contradiction. So x cannot be rational after all.

6. We have $x_1 = \frac{7}{4}$ and $x_1^2 - 3 = \frac{1}{16}$. So we put

$$x_2 = x_1 - \frac{1}{16 \times 2x_1} = x_1 - \frac{4}{32 \times 7} = x_1 - \frac{1}{56} = \frac{97}{56}.$$

Then

$$x_2^2 - 3 = \frac{9409}{3136} - 3 = \frac{1}{3136}$$

So we put

$$x_3 = x_2 - \frac{1}{3136 \times 2x_2} = \frac{97}{56} - \frac{28}{3136 \times 97} = \frac{97}{56} - \frac{1}{112 \times 97}$$

$$= \frac{97^2 \times 2 - 1}{112 \times 97} = \frac{18817}{112 \times 97}.$$

$$x_3^2 - 3 = \frac{354079489}{118026496} - \frac{354079488}{118026496} = \frac{1}{118026496}.$$

7.

a) $-1 \le -\dfrac{1}{n^3} < 0$ for all $n \in \mathbb{Z}_+$. So -1 is the mimimal element. There is no maximal element, because $-\dfrac{1}{(n+1)^3} > -\dfrac{1}{n^3}$ for all $n \in \mathbb{Z}_+$, so $\frac{1}{n^3}$ cannot be minimal for any $n \in \mathbb{Z}_+$.

b) $\{x \in \mathbb{R} : x^2 \le 5\} = [-\sqrt{5}, \sqrt{5}] = \{x \in \mathbb{R} : -\sqrt{5} \le x \le \sqrt{5}\}$. So $\sqrt{5}$ is the maximal element and $-\sqrt{5}$ is the minimal element.

c) $\{x \in \mathbb{R} : x^2 < 3\} = (-\sqrt{3}, \sqrt{3})$. This set has no maximal or minimal element because the numbers $\pm\sqrt{3}$ are not in the set, but yet for any real number $x$with $-\sqrt{3} < x < \sqrt{3}$, there are real numbers y and z with $-\sqrt{3} < y < x < z < \sqrt{3}$. For example one can take $y = (x - \sqrt{3})/2$ and $z = (x + \sqrt{3})/2$.

d) $A = \{x \in \mathbb{Q} : x^2 \le 3\} = \mathbb{Q} \cap [-\sqrt{3}, \sqrt{3}]$ Since $\pm\sqrt{3}$ are not rational,Acan also be written as $\mathbb{Q} \cap (-\sqrt{3}, \sqrt{3})$, and again has no maximal or minimal elements because although $-\sqrt{3} < x < \sqrt{3}$ for all $x \in A$, $\pm\sqrt{3} \notin A$ and yet for any $x \in A$ there are y and $z \in A$ with $y < x < z$. For example (although this much detail is not required) we can take $y = x - (3 - x^2)/4$ and $z = x + (3 - x^2)/4$.

e) $A = \mathbb{R}$ because $2 - x^2 \le 3 \Leftrightarrow x^2 \ge -1$, and this is true for all $x \in \mathbb{R}$. So there is no maximum element and no minimum element.

8. If $x \ge 1$ and $0 < \varepsilon < 1$ then

$$(x + \varepsilon)^2 = x^2 + 2x\varepsilon + \varepsilon^2 < x^2 + 3x\varepsilon$$

If in addition $x^2 < 56$ and $\varepsilon \le (5 - x^2)/2$ then $x^2 + 3x\varepsilon \le x^2 + 5 - x^2 = 5$ and hence $(x + \varepsilon)^2 < 5$.

a) $A = \{x \in \mathbb{Q} : x < 3\}$ is a Dedekind cut because for $y \in Q$, $y \ge 2 \Rightarrow y \notin A$ (which shows $A \ne \emptyset$ and $A \ne \mathbb{Q}$) and $y < x \Rightarrow y < 2 \Rightarrow y \in A$ and for $x \in A$, there is no maximal element: if $x < 2$ then $x < \frac{x+2}{2} < 2$, and $\frac{x+2}{2} \in \mathbb{Q}$.

b) In this case $A = \{x \in Q : x \ge -2\}$ which is not a Dedekind cut because it is not bounded above.

c) $A = \{x \in Q : x^2 < 3 \vee x < 1\}$.

- $3^2 = 9 > 3$, and $x > 3 \Rightarrow x^2 > 9 > 5$. So A is bounded above.
- If $x \in A$ and $y < x$ then either $y \leq 0 < 1$, in which case $y \in A$, or $0 \leq y < x$, in which case $y^2 < 5$ and again $y \in A$.
- If $x \in A$ with $x^2 < 5$, either $x < 1 \in A$ or $x \geq 1$ and we choose $\varepsilon \in \mathbb{Q}$ with $0 < \varepsilon < 5 - x^2)/3$. Then $x < x + \varepsilon$ and $x + \varepsilon \in A$. Hence x is not maximal in A for any $x \in A$ and there is no maximal element.

So A is a Dedekind cut.

d) $A = \{x \in \mathbb{Q} : x^2 < 5 \vee x > 1\}$ is not a Dedekind cut because (for example) $-3 \notin A$ but $0 \in A$

e) Again A is not a Dedekind cut because $-3 \notin A$ but $0 \in A$.

9. *Base case* If $A \subset \mathbb{R}$ has one element a then a is both a maximal and a minimal element of A.

Inductive step Now suppose inductively that any set with n elements has both a maximal element and a minimal element. Let A be a set with $n+1$ elements. Then there is a bijection $f : \{k \in \mathbb{Z}_+ : k \leq n+1\} \to A$. Let $B = \{f(k) : k \leq n\}$. Then by the inductive hypothesis B has a maximal element b_1 and a minimal element b_2, where $b_1 = f(i)$ for some $i \leq n$ and $b_2 = f(j)$ for some $j \leq n$. Then $A = B \cup \{f(n+1)$. There are three possibilities.

- $f(n+1) < b_2$. Then $f(n+1)$ is the minimum element of A and $b_1 = f(i)$ is the maximum element.
- $b_1 < f(n+1)$. Then $b_2 = f(j)$ is the minimum element of A and $f(n+1)$ is the maximum element.
- $n \geq 2$ and $b_2 < f(n+1) < b_1$. Then $b_2 = f(j)$ is the minimum element of A and b_1 is the maximum element.

So by induction a set with n elements has both a maximal and a minimal element for all $n \in \mathbb{Z}_+$.