MATH105 Solutions to Practice Problems 5

8. **Base case** $3^3 + 4^3 = 27 + 64 = 91 < 125 = 5^3$. So $3^n + 4^n < 5^n$ is true for $n = 3$.
Inductive step Now assume that $n \geq 3$ and $3^n + 4^n < 5^n$ and consider $3^{n+1} + 4^{n+1}$. We have

$$3^{n+1} + 4^{n+1} < 4 \cdot (3^n + 4^n) < 4 \cdot 5^n < 5^{n+1}$$

So

$$3^n + 4^n < 5^n \Rightarrow 3^{n+1} + 4^{n+1} < 5^{n+1}$$

Finishing By induction $3^n + 4^n < 5^n$ for all integers $n \geq 3$.

9.

$$\begin{array}{cc|c} 1 & 0 & 450 \\ 0 & 1 & 378 \end{array} \quad \begin{array}{c} R_1 - R_2 \\ \rightarrow \end{array} \quad \begin{array}{cc|c} 1 & -1 & 72 \\ 0 & 1 & 378 \end{array} \quad \begin{array}{c} \rightarrow \\ R_2 - 5R_1 \end{array} \quad \begin{array}{cc|c} 1 & -1 & 72 \\ -5 & 6 & 18 \end{array}$$

$$\begin{array}{c} R_1 - 4R_2 \\ \rightarrow \end{array} \quad \begin{array}{cc|c} 21 & -25 & 0 \\ -5 & 6 & 18 \end{array}$$

So the g.c.d. is $d = 18$, from the second row of the last matrix. Also from this second row we have

$$-5 \times 434 + 6 \times 364 = 14$$

so $a = -5$ and $b = 6$ The first row of the last matrix gives $21 \times 450 = 25 \times 378$. This number is 9450, and is the l.c.m..

10. We fix $n \in \mathbb{Z}_+$. Since $k \mid n! = \prod_{j=1}^{n} j$ for $2 \leq k \leq n$, it is also true that $k \mid n! + k$ for $2 \leq k \leq n$. Since k is divisible by at least one prime for any integer $k \geq 2$, and $n! + k > k$, it follows that $n! + k$ is not prime for any $2 \leq k \leq n$. There are $n - 1$ such numbers and hence they must all be contained in the same prime gap, which must have length $\geq n$.