MATH105 Solutions to Practice Problems

1. a) $m = 168$, $n = 63$.

$$\left(\begin{array}{cc|c} 1 & 0 & 168 \\ 0 & 1 & 63 \end{array}\right) \xrightarrow{R_1 - 2R_1} \left(\begin{array}{cc|c} 1 & -2 & 42 \\ 0 & 1 & 63 \end{array}\right) \xrightarrow[R_2 - R_1]{} \left(\begin{array}{cc|c} 1 & -2 & 42 \\ -1 & 3 & 21 \end{array}\right) \xrightarrow{R_1 - 2R_2} \left(\begin{array}{cc|c} 3 & -8 & 0 \\ -1 & 3 & 21 \end{array}\right)$$

So (i) $d = 21$, (ii) $a = -1$ and $b = 3$, that is, $-1 \times 168 + 3 \times 63 = 7$ and (iii) $m_1 = 8$ and $n_1 = 3$, that is, $168 = 8 \times 21$ and $63 = 3 \times 21$.

Using prime factorisation, $168 = 8 \times 21 = 2^3 \times 3 \times 7$, while $63 = 3^2 \times 7$. So the gcd is $3 \times 7 = 21$ and the lcm is $2^3 \times 3^2 \times 7 = 504$.

b) $m = 234$, $n = 416$.

$$\left(\begin{array}{cc|c} 1 & 0 & 234 \\ 0 & 1 & 416 \end{array}\right) \xrightarrow[R_2 - R_1]{} \left(\begin{array}{cc|c} 1 & 0 & 234 \\ -1 & 1 & 182 \end{array}\right) \xrightarrow{R_1 - R_2} \left(\begin{array}{cc|c} 2 & -1 & 52 \\ -1 & 1 & 182 \end{array}\right) \xrightarrow[R_2 - 3R_1]{} \left(\begin{array}{cc|c} 2 & -1 & 52 \\ -7 & 4 & 26 \end{array}\right)$$

$$\xrightarrow{R_1 - 2R_2} \left(\begin{array}{cc|c} 16 & -9 & 0 \\ -7 & 4 & 26 \end{array}\right)$$

So (i) $d = 26$, (ii) $a = -7$ and $b = 4$, that is, $-7 \times 234 + 4 \times 416 = 26$ and (iii) $m_1 = 9$ and $n_1 = 16$, that is, $234 = 9 \times 26$ and $416 = 16 \times 26$.

Using prime factorisation, $234 = 2 \times 117 = 2 \times 9 \times 13 = 2 \times 3^2 \times 13$, and $416 = 4 \times 104 = 4 \times 8 \times 13 = 2^5 \times 13$. So the gcd is $2 \times 13 = 26$ and the lcm is $2^5 \times 3^2 \times 13 = 3774$.

c) $m = 543$, $n = 1251$.

$$\left(\begin{array}{cc|c} 1 & 0 & 543 \\ 0 & 1 & 1251 \end{array}\right) \xrightarrow[R_2 - 2R_1]{} \left(\begin{array}{cc|c} 1 & 0 & 543 \\ -2 & 1 & 165 \end{array}\right) \xrightarrow{R_1 - 3R_2} \left(\begin{array}{cc|c} 7 & -3 & 48 \\ -2 & 1 & 165 \end{array}\right) \xrightarrow[R_2 - 3R_1]{} \left(\begin{array}{cc|c} 7 & -3 & 48 \\ -23 & 10 & 21 \end{array}\right)$$

$$\xrightarrow{R_1 - 2R_2} \left(\begin{array}{cc|c} 53 & -23 & 6 \\ -23 & 10 & 21 \end{array}\right) \xrightarrow{R_1 - 3R_2} \left(\begin{array}{cc|c} 53 & -23 & 6 \\ -182 & 79 & 3 \end{array}\right) \xrightarrow{R_1 - 2R_2} \left(\begin{array}{cc|c} 417 & -181 & 0 \\ -182 & 79 & 3 \end{array}\right)$$

So (i) $d = 3$, (ii) $a = -182$ and $b = 79$, that is, $-182 \times 543 + 79 \times 1251 = 4$ and (iii) $m_1 = 181$ and $n_1 = 417$, that is, $543 = 181 \times 3$ and $1251 = 417 \times 3$.

Using prime factorisation, $543 = 3 \times 181$, and 181 is prime. We can check this from the list of the first 1000 primes, or, alternatively, by checking that it is not divisible by 2, 3, 5, 7, 11 and 13. Also, $1251 = 9 \times 139 = 3^2 \times 139$, and 139 is prime. So 3 is the gcd, and $3^2 \times 139 \times 181 = 226431$ is the lcm.

2. $x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x - 1)(x + 1)(x^2 + 1)$. Putting $x = 10$, we have

$$10^4 - 1 = (10^2 + 1) \times (10 - 1) \times (10 + 1) = 101 \times 9 \times 11 = 101 \times 3^2 \times 11.$$

The last expression in the line above is the prime factorisation of $10^4 - 1 = 9999$.

3. We start by checking Binet's formula for $n = 1$ and $n = 2$:

$$u_1 = 1 = \frac{(1 + \sqrt{5}) - (1 - \sqrt{5})}{2\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1,$$

$$u_2 = \frac{((1 + \sqrt{5})^2 - (1 - \sqrt{5})^2}{4\sqrt{5}} = \frac{1 + 5 + 2\sqrt{5} - 1 - 5 + 2\sqrt{5}}{4\sqrt{5}} = \frac{4\sqrt{5}}{4\sqrt{5}} = 1.$$

Suppose the formula works for u_k for $1 \leq k \leq n$ where $n \geq 2$. Then

$$u_{n+1} = u_{n-1} + u_n = \frac{(1 + \sqrt{5})^{n-1} - (1 - \sqrt{5})^{n-1}}{2^{n-1}\sqrt{5}} + \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n\sqrt{5}}$$

$$= \frac{(1+\sqrt{5})^{n-1}(2+1+\sqrt{5})}{2^n\sqrt{5}} - \frac{(1-\sqrt{5})^{n-1}(2+1-\sqrt{5}}{2^n\sqrt{5}}$$

$$= \frac{(1+\sqrt{5})^{n-1}(6+2\sqrt{5}) - (1-\sqrt{5})^{n-1}(6-2\sqrt{5})}{2^{n+1}\sqrt{5}} = \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}}$$

because $6+2\sqrt{5} = (1+\sqrt{5})^2$ and $6-2\sqrt{5} = (1-\sqrt{5})^2$

So if the formula holds for $1 \leq k \leq n$, then it holds for $1 \leq k \leq n+1$. So by induction, Binet's formula holds for all $n \in \mathbb{N}$.

Now we compute u_n and $(1+\sqrt{5})^n 2^{-n}/\sqrt{5}$ for $n \leq 12$.

n	u_n	$(1+\sqrt{5})^n 2^{-n}/\sqrt{5}$	n	u_n	$(1+\sqrt{5})^n 2^{-n}/\sqrt{5}$
1	1	0.72360679775	7	13	12.98459713475
2	1	1.1708239325	8	21	21.0095149426
3	2	1.894427191	9	34	33.99411662902
4	3	3.06524758425	10	55	55.00363612328
5	5	4.95967477525	11	89	88.99775275231
6	8	8.0249223595	12	144	144.00138887561

The two quantities u_n and $(1+\sqrt{5})^n 2^{-n}/\sqrt{5}$ are getting closer. This is because $(1-\sqrt{5})^n 2^{-n}/\sqrt{5}$ is getting closer to 0. in fact for $n = 12$ this is 0.001388875..